CALCULUS

F. M. AND C.

AN ELEMENTARY TREATISE

ON

CALCULUS

A TEXT BOOK FOR

COLLEGES AND TECHNICAL SCHOOLS

BY

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PREFACE.

1. We believe that the most important thing in the teaching of calculus is to lead the student to a clear understanding of principles. Therefore our chief endeavor has been to develop the subject as simply and as directly as possible.

2. We fully appreciate the importance of extensive practice in the handling of algebraic transformations. Therefore we have given an adequate collection of formal problems in differentiation and integration.

3. We are convinced, however, that it is a pedagogical mistake, in a text book on calculus for young men, to break the thread of the textual discussion by unnecessary algebraic developments and by large and frequent groups of purely formal problems. We believe indeed that the proverbial unintelligibility of calculus is to a very great extent a psychological consequence of this almost universal and really hideous feature. Therefore we have arranged the greater portion of our formal problems in an appendix.

4. Nearly every elementary science text in existence carries a false suggestion of completeness and finality, and there are two things which young men should understand in connection with their study of the mathematical sciences. The first is that such study is exacting beyond all compromise, involving as it does a degree of constraint which it is beyond the power of any teacher greatly to mitigate. And the second is that the completest science stands abashed before the infinitely complicated and fluid array of phenomena of the material world, except only in the assurance which method gives. We hope that this elementary treatise on calculus may prove to be sufficiently definite and exacting to be useful; next to this there is nothing we could wish to have more in evidence from beginning to end than its incompleteness.

In order to emphasize the incomplete character of this text we have introduced references to more complete treatises throughout the text and we have given in Appendix *C* a carefully selected list of treatises on mathematics and on mathematical physics. Teachers who use this text should, we believe, direct the students' attention to this appendix.

In our brief reference to the infinitesimal method in article 20 we do not wish to be thought of as taking sides in the old controversy as between the "method of limits" and the "method of infinitesimals" in calculus. Article 20 is unquestionably fallacious as it stands; and the same may be said of the discussion of divergence and curl in articles 126 and 129. Indeed articles 20, 126 and 129 may be characterized as mere plausibilities, and the harder one tries to understand them the more vague and unintelligible they become. The fact remains, however, that the infinitesimal method contributes very greatly to directness and simplicity of speech in the discussion of physical problems, and the idea of infinitesimals is therefore used throughout this text. Any one who is disturbed by the element of easy plausibility that is involved in the s raight-forward use of the infinitesimal method in the discussion of physical problems should heed the advice given by D'Alembert to a young student, "Go ahead, young man, go ahead! Conviction will come to you later."

"The absolute requisites for the study of this work are: a knowledge of elementary algebra to the binomial theorem (according to the usual arrangement), plane and solid geometry, trigonometry, and the most simple parts of the usual applications of algebra to geometry."

This was said by De Morgan in the preface to his great treatise on *Differential and Integral Calculus* (London, 1842), in comparison with which this book is a primer.

FRANKLIN, MACNUTT and CHARLES.

SOUTH BETHLEHEM, PA., March 22, 1913.

CALCULUS.

In the study of phenomena which depend upon conditions which vary in time, that is, upon conditions which vary from instant to instant, it is necessary to direct the attention to what is taking place at an instant; or, in other words, to direct the attention to what takes place during a very short interval of time; or, borrowing a phrase from the photographer, to make a *snap-shot*, as it were, of the varying conditions. In the study of phenomena which depend upon conditions which vary from point to point in space, the attention must be directed to what takes place in a very small region.

This paying attention to what takes place during a very short interval of time or in a very small region of space does not refer to *observation* but to *thinking*, it is a mathematical method and it is called calculus. Thus the density of a body is its mass divided by its volume. This definition, and indeed the idea of density, applies only to a homogeneous substance. To apply the idea of density to a non-homogeneous substance one must think of a very small portion of the substance. The same thing is true of every measurable property of a substance. Thus the idea of elastic strain as a measurable effect is easily established when every part of a body is similarly strained as in the case of a stretched rod, but to apply this precise idea of strain to a bent beam or to a twisted rod one must think of a small portion of the beam or rod.

Two distinct methods are involved in the directing of the attention to what takes place during very short intervals of time or in very small regions of space, as follows:

(*a*) *The method of differential calculus.* A phenomenon may be prescribed as a pure assumption, and the successive instantaneous

aspects may then be derived from this prescription. Thus we may prescribe uniform motion of a particle in a circular path, and then determine the acceleration of the particle at each instant.

Or, a condition in space may be prescribed as a pure assumption, and the minute aspects may then be derived from this prescription. Thus we may prescribe the distribution of temperature along a rod, and then determine the temperature gradient at each point from this prescription.

(*b*) *The method of integral calculus.* It frequently happens that we know and can easily formulate the action which takes place at a given instant or in a small region of space. The problem then is to build up an idea of the result of this action throughout a finite interval of time or throughout a finite region of space. Thus the acceleration of a body may be known at each instant, and from this knowledge we may find the velocity gained and the distance traveled in a finite interval of time. Or, consider a disk rotating at a given speed. It is easy to establish a formula for the energy of a very small particle of this rotating disk, and it is then possible to derive an expression for the energy of the entire disk.

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"In view of the acknowledged difficulty of calculus the student must be willing to stop in his course until he can form exact notions and acquire precise ideas."

AUGUSTUS DE MORGAN.

CHAPTER I.

GENERAL SURVEY OF DIFFERENTIAL AND INTEGRAL CALCULUS.

1. Constant quantities and variable quantities.—Elementary algebra deals only with quantities which do not change in value (*constant quantities*). For example let x be the quantity which subtracted from 12 leaves a remainder equal to $\frac{1}{2}x$. That is, $12 - x = \frac{1}{2}x$, whence $x = 8$. The quantity x has the same value throughout this discussion and it is therefore called a constant quantity. Calculus is a branch of algebra and it deals not only with constant quantities but also and especially with quantities which change in value (*variable quantities*). Thus one of the simplest problems of calculus is to consider how rapidly x^2 grows in value when x is imagined to grow at a definite rate.

Variable quantities fall into three fairly distinct classes, namely: (*a*) Quantities which vary in time, (*b*) quantities which vary in space, and (*c*) quantities which are arbitrarily assumed to vary. An example of the first class is the varying temperature of the air *at a given place* as the seasons come and go. An example of the second class is the varying temperature of the air from place to place *at a given time*.

Variables of the first and second classes are sometimes called *natural variables* to distinguish them from variables of the third class, examples of which are considered in Arts. 10 and 11.

2. Value at a given instant. Value at a given point.—Let y represent the amount of water in a pail into which a small stream of water is flowing. Evidently y is a changing quantity. If the flow of water were to be shut off at *any given instant* there would be a *definite quantity* of water left in the pail, and, therefore, there is a definite amount of water in the pail at each instant even while

the inflow continues. *Any changing quantity y has a definite value at each instant.**

A quantity which varies in space has a definite value at each point.* Thus everyone understands that there is a definite temperature at Philadelphia although it may be different from the temperature at New York or Washington. An iron rod with one end in the fire has a definite temperature at each point.

3. Increments and decrements.—It is frequently desirable to consider an increase (or a decrease) in the value of a variable quantity. Such an increase is called an *increment*. A decrease is called a *decrement*. For example, let y be a variable quantity. An increment of y is usually represented by the symbol Δy. When Δy is negative it is a decrement. This symbol Δy *does NOT mean* Δ *multiplied by* y, **it is a single symbol**, and it may be read *increment-of-y* or *delta-y*. The latter is preferable because of its brevity. The Greek letter Δ used as a prefix always means *increment of*.

4. Rate of change of a quantity which varies in time.—Consider a pail into which water is flowing in a small stream. Let y be the amount of water in the pail. Then y is evidently a changing quantity. Consider the amount of water that flows into the pail during a short interval of time Δt; this amount of water is evidently the increment of y during the short interval of time, and it is to be represented by the symbol Δy. The quotient $\dfrac{\Delta y}{\Delta t}$ is called the *average rate of change of y during the interval Δt*.†

If the inflowing stream of water is constant, say, always 2 cubic inches of water per second, then if Δt is chosen smaller and smaller, the value of Δy will be *proportionately* smaller and smaller, and the value of the quotient $\dfrac{\Delta y}{\Delta t}$ will be exactly 2 cubic inches per second whatever the duration or length of the time-interval Δt.

* This may not be true of a discontinuous variable.

† Thus if y is the amount of money a man has saved, then, if he saves \$20 more during 30 days, we have $\Delta y = \$20$ and $\Delta t = 30$ days, and the average rate of saving during the 30 days will be $66\frac{2}{3}$ cents per day.

If the inflowing stream is variable, then the value of the quotient $\frac{\Delta y}{\Delta t}$ will not be the same for different values of Δt, but the amount of water which flows into the pail *during a very short interval of time* will be very nearly proportional to the interval. For example, a certain amount of water W flows into the pail during a particular one-thousandth of a second. Imagine this particular one-thousandth of a second to be divided in halves; then only a very little more than $\frac{1}{2}W$ will flow into the pail during one of the half-thousandths of a second and a very little less than $\frac{1}{2}W$ will flow into the pail during the other half-thousandth of a second. *If a shorter and shorter interval of time be taken* **the amount of inflow of water will be more and more nearly in EXACT proportion to the duration of the interval.** This means that the quotient $\frac{\Delta y}{\Delta t}$ approaches a definite limiting value as Δt and Δy both approach zero; and this limiting value of $\frac{\Delta y}{\Delta t}$ is called *the rate of change of* y *at a certain instant* or **the instantaneous rate of change of y.**

The limiting value of $\frac{\Delta y}{\Delta t}$ is always represented by $\frac{dy}{dt}$ which means *the rate of change of* y *at a given instant.*

Nearly everyone falls into the idea that such an expression as 10 feet per second means 10 feet of actual movement in an actual second of time, but a body moving at a velocity of 10 feet per second might not continue to move for a whole second or its velocity might change before a whole second has elasped. Three cubic inches per second is the same rate of inflow of water into a pail as 2,070 cubic yards per year, but to specify the *rate* of inflow in cubic yards per year does not mean that a whole cubic yard of water flows into the pail nor does it mean that the inflow continues for a year. A man does not need to work for a whole month to earn money at the *rate* of \$60.00 per month, nor for a whole day to earn money at the *rate* of \$2.00 per day. A falling body has

a velocity of 19,130,000 miles per century after it has been falling for one second, but to specify its velocity in miles per century does not mean that it falls as far as a mile nor that it continues to fall for a century! The units of length and time which appear in the specification of a velocity are completely swallowed up, as it were, in the idea of velocity; and the same thing is true of the specification of any rate.

5. Continuous variables and discontinuous variables.—A quantity which changes by sudden jumps is called a *discontinuous variable*. For example the amount of money one has is a discontinuous variable, because a debt is made on the instant that one decides to accept a purchase; that is to say, money is spent in lumps, and a lump of money is spent during an indefinitely short time. *The amount of money spent during an interval of time does NOT become more and more nearly proportional to the interval as the interval grows shorter and shorter*, and consequently it is meaningless to speak of *the rate of spending money at a given instant*. If y is a discontinuous variable and if the time-interval Δt happens to include a jump in the value of y, then the value of Δy remains finite as Δt approaches zero and the quotient $\frac{\Delta y}{\Delta t}$ approaches infinity. The rate of change of a discontinuous variable at a given instant is unthinkable.

The amount of water in a pail, as considered in Art. 4, is an example of what is called a *continuous variable*. If y is a continuous variable and if Δy is the increment of y during the time-interval Δt, then Δy becomes more and more nearly proportional to Δt as Δt approaches zero; that is to say, the quotient $\frac{\Delta y}{\Delta t}$ approaches a definite limiting value as Δt approaches zero, and this limiting value of $\frac{\Delta y}{\Delta t}$ is called the instantaneous rate of change of y.

In the study of calculus we deal almost exclusively with continuous variables.

6. Example showing determination of instantaneous rate of change.—Let us *assume* that the amount of water in the pail in the discussion of Art. 4 is proportional to the square of the time t which has elapsed since a chosen instant. Then we may write

$$y = kt^2 \tag{1}$$

where k is a constant. A moment later t has increased and of course y has increased also. Let us represent the new value of t by $t + \Delta t$ and the new value of y by $y + \Delta y$. Then, since equation (1) is assumed to be true for all values of t and y, we have

$$y + \Delta y = k(t + \Delta t)^2.$$

or

$$y + \Delta y = kt^2 + 2kt \cdot \Delta t + k(\Delta t)^2 \tag{2}$$

Subtracting equation (1) from equation (2) member by member, we have:

$$\Delta y = 2kt \cdot \Delta t + k(\Delta t)^2 \tag{3}$$

Whence, dividing both members by Δt, we have:

$$\frac{\Delta y}{\Delta t} = 2kt + k \cdot \Delta t \tag{4}$$

Now it is evident that $k \cdot \Delta t$ approaches zero as Δt approaches zero, and therefore the limiting value of $\frac{\Delta y}{\Delta t}$ (as Δt approaches zero) is $2kt$. Hence, writing $\frac{dy}{dt}$ for the limiting value of $\frac{\Delta y}{\Delta t}$, we have:

$$\frac{dy}{dt} = 2kt \tag{5}$$

That is, the rate of change of y is at each instant equal to $2kt$.

Observation or thinking; which?—The discussion, in Art. 4, of the rate of increase of the amount of water in a pail presents a serious difficulty. How is one to know the increment of water which occurs during an indefinitely short interval of time? One

certainly cannot measure it. Observation and measurement are entirely useless in the consideration of an indefinitely small change which takes place during an indefinitely short interval of time. One cannot *observe* such things, one can only *think* about them. Indeed the whole matter at present under discussion is a matter of mathematical reasoning.

7. Gradient of a quantity which varies in space.—Consider an iron rod which stands with one end in a fire. Then the temperature of the rod varies from point to point along the rod. Consider two points very near together, let ΔT be their difference in temperature and let Δx be their distance apart. The quotient $\frac{\Delta T}{\Delta x}$ approaches a definite limiting value when Δx is chosen smaller and smaller, and this limiting value of $\frac{\Delta T}{\Delta x}$ is called the *temperature gradient* at the point (degrees per inch). In this example T varies *continuously* along the rod, it does not vary by jumps, otherwise the gradient of T at a point would be unthinkable.

The limiting value of the quotient $\frac{\Delta T}{\Delta x}$ is always represented by $\frac{dT}{dx}$, and it means the gradient of T at a given point.

Another quantity which varies from point to point in space is the pressure of the atmosphere; the higher one goes the less the pressure. Consider two points A and B, one of which is at a distance Δx above the other, and let Δp be the amount by which the pressure at A exceeds the pressure at B. Then $\frac{dp}{dx}$, the *limiting value of the quotient* $\frac{\Delta p}{\Delta x}$, is called the *pressure gradient* at A.* The pressure gradient of the atmosphere at sea level is usually about 2.3 pounds-per-square-inch per mile.

The word *gradient* as used in the above discussion comes from the word *grade* meaning the steepness of a slope. Thus if the

* Or at B; the two points A and B approach coincidence as Δx approaches zero.

slope of a hill at a given place rises Δh feet in a horizontal distance of Δx feet, then the limiting value of $\frac{\Delta h}{\Delta x}$ when Δx is made smaller and smaller is called the grade of the hill at the given place. The grade of a hill at a point might be one foot of rise per ten feet of horizontal distance, or ten feet of rise per hundred feet of horizontal distance, or 10 per cent. as it is usually expressed. It is evident that 10 feet of rise per 100 feet of horizontal distance does not mean necessarily 10 feet of actual rise in 100 feet of horizontal distance, because the slope may not be 100 feet long and the grade may vary from point to point.

Similarly a pressure gradient of 2.3 pounds-per-square-inch per mile, which is the upward gradient of atmospheric pressure at sea level, does not mean that the atmospheric pressure at a point one mile above sea level is 2.3 pounds per square inch less than at sea level, because the pressure gradient becomes less and less at points higher and higher above the sea. The units which appear in the specification of a grade or a gradient are completely swallowed up, as it were, in the idea of grade or gradient just as the units which appear in the specification of a rate are swallowed up in the idea of rate.

8. Example showing the determination of a gradient at a point. —Consider a metal bar AB, Fig. 1. Suppose the temperature

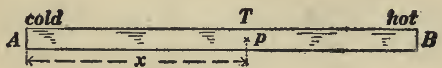


Fig. 1.

of the bar to be zero at the end A, and let us *assume* the temperature T at the point p to be:

$$T = kx^2 \tag{1}$$

where x is the distance from A to p, and k is a constant. Under these conditions let it be required to find the temperature gradient at the point p. Now equation (1) is true for every value of x and T. Therefore writing $x + \Delta x$ for x and writing

$T + \Delta T$ for T we have

$$T + \Delta T = k(x + \Delta x)^2 \tag{2}$$

or

$$T + \Delta T = kx^2 + 2kx\cdot\Delta x + k(\Delta x)^2 \tag{3}$$

Subtracting equation (1) from equation (3) member by member, we have:

$$\Delta T = 2kx\cdot\Delta x + k(\Delta x)^2 \tag{4}$$

whence

$$\frac{\Delta T}{\Delta x} = 2kx + k\cdot\Delta x \tag{5}$$

Now $k\cdot\Delta x$ approaches zero as Δx approaches zero, and it is therefore evident that the limiting value of $\frac{\Delta T}{\Delta x}$ is $2kx$. Therefore, writing $\frac{dT}{dx}$ for the limiting value of $\frac{\Delta T}{\Delta x}$, we have:

$$\frac{dT}{dx} = 2kx \tag{6}$$

It is evident from this equation that the temperature gradient is zero at the end A of the bar where $x = 0$; and it is evident that the temperature gradient grows greater and greater (steeper and steeper) at points farther and farther from A where x is larger and larger.

PROBLEMS.

1. The amount of water in a pail is assumed to be proportional to the square of the time which has elasped since a chosen instant, according to equation (1) of Art. 6. It is evident from this equation that the amount of water in the pail is zero when $t = 0$. After the lapse of 20 seconds there is 600 cubic inches of water in the pail. Find the value of k and state the unit in terms of which k is expressed. Ans. 1.5 cubic inches per second per second.

2. Using the value of k from problem 1, find the rate at which water flows into the pail, (a) when $t = 5$ seconds, and (b) when $t = 20$ seconds. Ans. (a) 15 cubic inches per second, (b) 60 cubic inches per second.

3. Find the average rate of inflow of water into the pail during the time from $t = 0$ to $t = 20$ seconds. Ans. 30 cubic inches per second.

4. The temperature of an iron rod increases along the rod in accordance with equation (1) of Art. 8. At a point 20 inches from the end of the rod ($x = 20$ inches), the value of T is 240° C. Find the value of k and state the unit in terms of which k is expressed. Ans. 0.6 degree per inch per inch.

5. Using the value of k from problem 4, find the temperature gradient (*a*) at the point where $x = 5$ inches and (*b*) at the point where $x = 20$ inches. Ans. (*a*) 6 degrees per inch, (*b*) 24 degrees per inch.

6. (*a*) Find the average temperature gradient along the iron rod of problem 4 between $x = 0$ and $x = 20$ inches. (*b*) Find the average temperature gradient between $x = 10$ inches and $x = 11$ inches. Ans. (*a*) 12 degrees per inch, (*b*) 12.6 degrees per inch.

9. Arbitrary variations.—One of the most important things in calculus is to consider how fast such an expression as x^2 or x^3 or $\sin x$ changes when x is arbitrarily assumed to grow steadily in value. Such variations may be called *arbitrary variations*. The very great importance of arbitrary variations will be understood when the student reaches Arts. 22 and 23.

10. A purely algebraic example of an arbitrary variation.—Having given the equation

$$y = ax^2 \tag{1}$$

let it be required to find the limiting value of the quotient $\frac{\Delta y}{\Delta x}$ when the arbitrary increment of x, namely Δx, approaches zero. Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (1) we have

$$y + \Delta y = a(x + \Delta x)^2 \tag{2}$$

or

$$y + \Delta y = ax^2 + 2ax \cdot \Delta x + a(\Delta x)^2 \tag{3}$$

whence, subtracting equation (1) from equation (3) member by

memoer, we have

$$\Delta y = 2ax \cdot \Delta x + a(\Delta x)^2 \tag{4}$$

and dividing both members by Δx we have

$$\frac{\Delta y}{\Delta x} = 2ax + a \cdot \Delta x \tag{5}$$

But $a \cdot \Delta x$ approaches zero when Δx approaches zero, and therefore it is evident that the limiting value of $\frac{\Delta y}{\Delta x}$ is $2ax$. The limiting value of $\frac{\Delta y}{\Delta x}$ is always represented by $\frac{dy}{dx}$. Therefore we have:

$$\frac{dy}{dx} = 2ax \tag{6}$$

11. Two more purely algebraic examples of arbitrary variations.—(*a*) Having given the equation

$$y = ax^3 \tag{1}$$

let it be required to find the limiting value of $\frac{\Delta y}{\Delta x}$ when the arbitrary increment of x approaches zero. Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (1) we have

$$y + \Delta y = a(x + \Delta x)^3 \tag{2}$$

or

$$y + \Delta y = ax^3 + 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{3}$$

whence, subtracting equation (1) from equation (3) member by member, we have

$$\Delta y = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{4}$$

or

$$\frac{\Delta y}{\Delta x} = 3ax^2 + 3ax \cdot \Delta x + a(\Delta x)^2 \tag{5}$$

Now $3ax \cdot \Delta x$ and $a(\Delta x)^2$ both approach zero when Δx approaches zero, and therefore it is evident that the limiting value of $\frac{\Delta y}{\Delta x}$ is

$3ax^2$; that is,

$$\frac{dy}{dx} = 3ax^2 \tag{6}$$

(b) Having the equation

$$y = \frac{a}{x} \tag{7}$$

let it be required to find the limiting value of $\frac{\Delta y}{\Delta x}$ when the arbitrary increment of x approaches zero. Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (7) we have

$$y + \Delta y = \frac{a}{x + \Delta x} \tag{8}$$

Subtracting equation (7) from equation (8) member by member, we have

$$\Delta y = \frac{a}{x + \Delta x} - \frac{a}{x} \tag{9}$$

Reducing the two fractions to a common denominator we have:

$$\Delta y = - \frac{a \cdot \Delta x}{x^2 + x \cdot \Delta x} \tag{10}$$

whence

$$\frac{\Delta y}{\Delta x} = - \frac{a}{x^2 + x \cdot \Delta x} \tag{11}$$

but $x^2 + x \cdot \Delta x$ approaches x^2 as its limit when Δx approaches zero, and therefore the limiting value of $\frac{\Delta y}{\Delta x}$ is $-\frac{a}{x^2}$; that is:

$$\frac{dy}{dx} = - \frac{a}{x^2} \tag{12}$$

The derivative $\frac{dy}{dx}$ is sometimes called the rate of change of y with respect to x.

PROBLEMS.

The following problems can be solved by the methods employed in Arts. 10 and 11. In each case find the rate of change of y with respect to x.

1. $y = ax$, $\qquad \dfrac{dy}{dx} = a.$

2. $y = ax^5$, $\qquad \dfrac{dy}{dx} = 5ax^4.$

3. $y = mx^2 + n$, $\qquad \dfrac{dy}{dx} = 2mx.$

4. $y = ax^2 + bx + c$, $\qquad \dfrac{dy}{dx} = 2ax + b.$

5. $y = ax^3 + bx^2$, $\qquad \dfrac{dy}{dx} = 3ax^2 + 2bx.$

6. $y = \dfrac{a}{x^2}$, $\qquad \dfrac{dy}{dx} = -\dfrac{2a}{x^3}.$

7. $y = \dfrac{a}{x^3}$, $\qquad \dfrac{dy}{dx} = -\dfrac{3a}{x^4}.$

8. $y = \dfrac{a}{bx^2 + x}$, $\qquad \dfrac{dy}{dx} = \dfrac{-a(2bx + 1)}{(bx^2 + x)^2}.$

9. $y = \dfrac{3x}{x + 3}$, $\qquad \dfrac{dy}{dx} = \dfrac{9}{(x + 3)^2}.$

10. $y = \dfrac{ax}{b - x}$, $\qquad \dfrac{dy}{dx} = \dfrac{ab}{(b - x)^2}.$

11. $y = \dfrac{2x - 5}{x + 2}$, $\qquad \dfrac{dy}{dx} = \dfrac{9}{(x + 2)^2}.$

12. $y = \dfrac{x}{(x - 1)^2}$, $\qquad \dfrac{dy}{dx} = -\dfrac{x + 1}{(x - 1)^3}.$

12. Functions.—When a spring is subjected to a stretching force F, a certain elongation e is produced as shown in Fig. 2. *For*

every value of e *there is a definite corresponding value of* F. When two quantities are associated in this way either one is said to be a

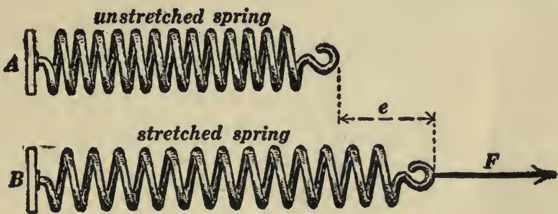


Fig. 2.

The elongation of the spring is a function of the stretching force.

function of the other. Thus the elongation e produced by a given force F in Fig. 2 is a function of F; or the force F required to produce a given elongation e is a function of e. The

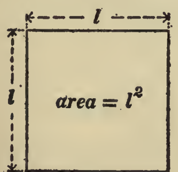


Fig. 3.

The area of a square is a function of length of side.

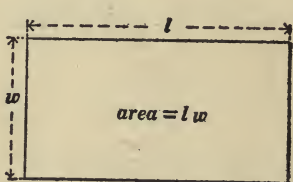


Fig. 4.

The area of a rectangle is a function of length and width.

meaning of the word function is further illustrated by Figs. 3, 4, 5, 6, 7 and 8.

13. Dependent and independent variables.—If l in Fig. 3 is assumed to be arbitrarily variable it is called an *independent variable*, and the area of the square is called a *dependent variable*, because its value is determined or fixed when the value of l is given. The quantity of air pumped into the cylinder in Fig. 7 may be as much as one pleases, the temperature of the cylinder

and air may be increased or decreased at will, and the volume of the cylinder may be changed by moving the piston. Therefore

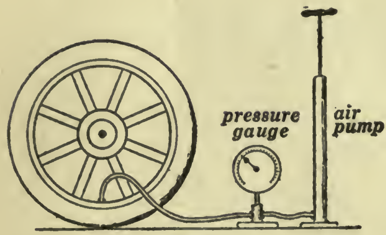


Fig. 5.

The pressure of the air is a function of the amount of air pumped into the tire.

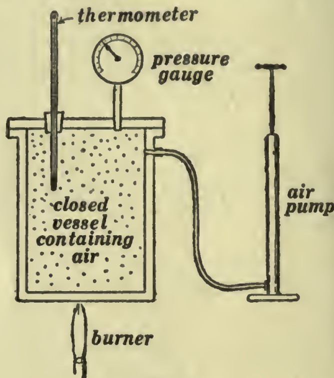


Fig. 6.

The pressure of the air in the vessel is a function of the amount of air pumped into the vessel and of the temperature.

the three quantities, (*a*) quantity of air, (*b*) temperature of the air, and (*c*) volume of space occupied by the air are called *independent variables,* and the pressure of the air is called a *dependent variable,*

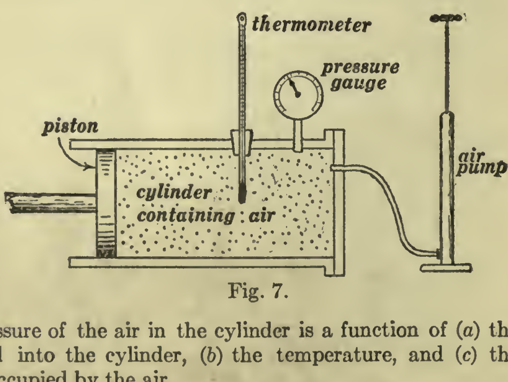


Fig. 7.

The pressure of the air in the cylinder is a function of (*a*) the amount of air pumped into the cylinder, (*b*) the temperature, and (*c*) the volume of the space occupied by the air.

because its value is fixed or determined when the quantities (a), (b) and (c) are given.

From this discussion it is evident that a dependent variable may be a function of one, two, three or more independent variables.

14. Natural functions and artificial functions.—We have heretofore taken the point of view that two quantities must be connected physically if either is to be a function of the other. Such a function may be called a *natural function*. It is, however, of very great importance to consider functional relations which grow out of or are expressed by algebraic equations. Such functions may be called *artificial functions*. Some idea of the importance of artificial functions may be obtained by looking back at Articles 4 and 6, and 7 and 8. Natural functions can be in many cases expressed algebraically; indeed no function can be handled in calculus unless the function is expressed algebraically.

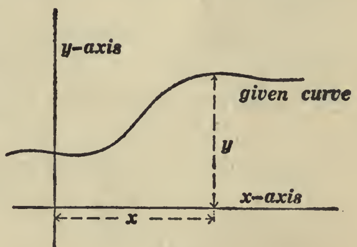


Fig. 8.

The ordinate of a point on a curve is a function of the abscissa of the point.

15. Tabulation of a function.—When one quantity is a function of another it is often convenient to express the functional relation by means of a table giving pairs of corresponding values of the two quantities. An ordinary table of logarithms is such a table, a table of sines or cosines is such a table, the mortality table* used in life insurance is such a table.

When a functional relation is determined by experiment the

* If one were to consider all men in the United States who are now forty years of age and keep a future record of their deaths one would find that their average age at death would be, say, sixty-five years. This is expressed by saying that the expectancy of life at forty years is twenty-five years, because on the average a man forty years old may expect to live twenty-five years. The expectancy of life is, of course, less and less with increasing age, and there is a definite expectancy of life corresponding to each age. That is, expectancy of life is a function of age.

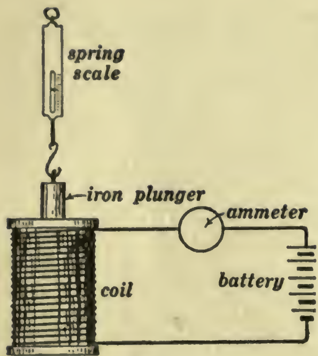


Fig. 9.

observed quantities are always arranged in tabular form. For example Fig. 9 represents an ammeter arranged to measure the electric current flowing through a coil of wire, and a spring scale arranged to measure the force with which an iron plunger is pulled into the coil; and the accompanying table shows a series of observed values of current in amperes and the corresponding pulls in pounds.

OBSERVED RELATION BETWEEN CURRENT AND PULL IN FIG. 9.

Current in Amperes.	Pull on Plunger in Pounds.
1.0	*3.9*
2.0	*23.5*
3.0	*40.0*
4.0	*48.1*

16. Graphical representation of a function.—The relation between the pull of the spring and the reading of the ammeter in Fig. 9, as shown in the table of Art. 15, may be represented graphically by a curve of which the abscissas represent ammeter readings and of which the ordinates represent corresponding readings of the spring scale. Such a curve is shown in Fig. 10. In the same way an algebraic function may be represented graphically. Thus the curve in Fig. 11 represents the function $y = x^3$.

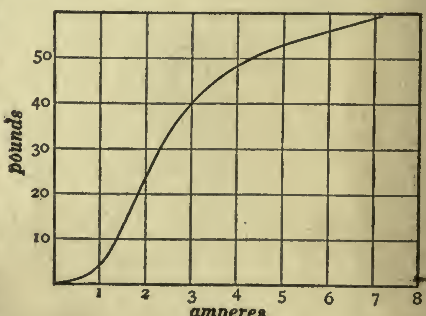


Fig. 10.

17. Derivative of a function.—Let y be a function of x and let Δy be the increment of y due to an arbitrary increment of x. Then the limiting value of the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is called *the derivative of* y *with respect to* x or *the rate of change of* y *with respect to* x.*

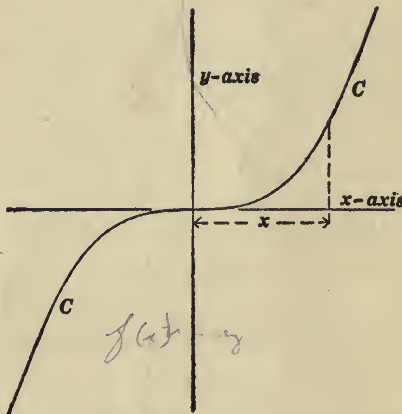


Fig. 11.

When one wishes to speak in general terms of any function of x and its derivative, the function may be represented by $f(x)$ or by $\phi(x)$ and the derivative of the function with respect to x may be represented by $f'(x)$ or by $\phi'(x)$.

Meaning of a derivative.—Let y be a function of x and let x be assumed to grow steadily at a definite rate, *then* y *grows* $\frac{dy}{dx}$ *times as fast as* x *at each instant.* For example, let $y = x^2$; then $\frac{dy}{dx} = 2x$, according to Art. 10, and the following relations exist:

y increases $2x(= 20)$ times as fast as x while x is passing through the value $x = 10$,

y increases $2x(= 22)$ times as fast as x while x is passing through the value $x = 11$,

y increases $2x(= 24)$ times as fast as x while x is passing through the value $x = 12$,

etc., etc., etc.

As another example, let $y = \frac{1}{x}$; then $\frac{dy}{dx} = -\frac{1}{x^2}$, according to Art. 11, and the following relations exist:

* Sometimes also called *the differential coefficient of* y *with respect to* x.

y *decreases* $\frac{1}{x^2}\left(=\frac{1}{4}\right)$ as fast as x *increases* when x is passing through the value 2,

y *decreases* $\frac{1}{x^2}\left(=\frac{1}{9}\right)$ as fast as x *increases* when x is passing through the value 3,

etc., etc., etc.

A negative value of $\frac{dy}{dx}$ shows that y decreases as x increases.

The derivative of a function of x is itself a function of x.—If $y = ax^2$, then $\frac{dy}{dx} = 2ax$, according to Art. 10. In this case it is evident that $\frac{dy}{dx}$ is a function of x because it is equal to $2ax$ and it has a definite value for every value of x. If $f(x)$ represents any function of x, then its derivative $f'(x)$ is in general a function of x also.

18. Graphical representation of derivative. Slope of a curve at a point.—Consider the curve CC, Fig. 12. This curve represents

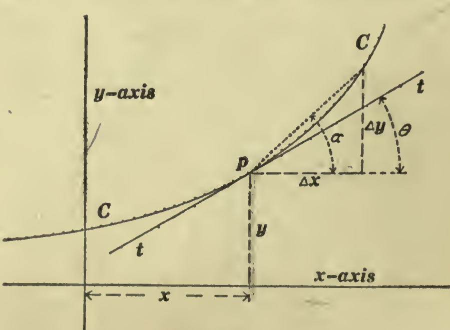


Fig. 12.

a definite function, that is, the ordinate y is a definite function of the abscissa x, the point p being anywhere on the curve. If x is increased by the amount Δx, then y will be increased by the

amount Δy as shown, and the ratio $\frac{\Delta y}{\Delta x}$ is equal to the tangent of the angle α.

Draw the line tt touching the curve at p as shown. As Δx approaches zero the angle α becomes more and more nearly equal to the angle θ, in fact θ is the limiting value of α, and, of course, the limiting value of $\frac{\Delta y}{\Delta x}$ is $\frac{dy}{dx}$. Therefore, since $\tan \alpha = \frac{\Delta y}{\Delta x}$, we must have $\tan \theta = \frac{dy}{dx}$. That is, *the function y being represented by the ordinates of a curve, the derivative $\frac{dy}{dx}$ is represented by the steepness or grade of the curve at each point.*

19. Derivative notation and differential notation.—Consider the function

$$y = ax^2 \tag{1}$$

of which the derivative is:

$$\frac{dy}{dx} = 2ax \tag{2}$$

Heretofore we have looked upon $\frac{dy}{dx}$ as a single symbol the meaning of which is explained in Art. 17. Convenience of notation sometimes makes it desirable to write equation (2) thus:

$$dy = 2ax \cdot dx \tag{3}$$

This equation expresses the limiting relation between the increments Δy and Δx, that is to say, the relation which is approached as Δy and Δx both approach zero; dy is called the *differential of* y (that is to say, the differential of ax^2, because y here stands for ax^2), and dx is called the *differential of* x. These two differentials may be thought of as indefinitely small increments of y and x respectively.*

* When $y = ax^2$ we have

$$\Delta y = 2ax \cdot \Delta x + a(\Delta x)^2$$

according to Art. 10. That is to say, the increment of y is not equal to $2ax$ times the increment of x unless both increments are indefinitely small.

Or, if one wishes to think of dy and dx as having a physical meaning they may be thought of as abbreviated expressions for $\frac{dy}{dt}$ and $\frac{dx}{dt}$ respectively, that is, dy may be thought of as *the rate of change of* y and dx may be thought of as *the rate of change of* x.

PROBLEMS.

1. A pail is 10 inches in diameter, so that the volume of water in the pail is $y = \frac{100\pi}{4} x$, where x is the depth of the water in the pail. Find how fast water must be poured into the pail to cause the water level to rise at a velocity of 4 inches per second. Ans. 100π cubic inches per second.

2. Water flows at a constant rate of 30 cubic inches per second into a metal cone of which the dimensions are as shown in Fig. *p*2. Find the velocity at which the water level rises (*a*) when $x = 5$ inches, and (*b*) when $x = 15$ inches. Ans. (*a*) 2.715 inches per second, (*b*) 0.302 inch per second. answers are [illegible]

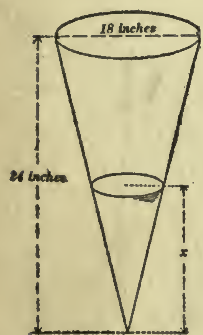


Fig. *p*2.

3. The sides of a square are growing at the rate of 5 inches per second. Find the rate of growth of the area of the square (*a*) when the sides of the square are 10 inches and (*b*) when the sides of the square are 20 inches. Ans. (*a*) 100 square inches per second, (*b*) 200 square inches per second.

4. The radius of a circle is growing at the rate of 5 inches per second. Find the rate of growth of the area of the circle (*a*) when the radius is 10 inches and (*b*) when the radius is 20 inches. Ans. (*a*) 100π square inches per second, (*b*) 200π square inches per second.

5. The edges of a cube are growing at the rate of 5 inches per second. Find the rate of growth of the volume of the cube when the edges are 10 inches long. Ans. 1,500 cubic inches per second.

6. Air is blown into a soap bubble at the rate of 10 cubic inches per second. Find the rate of increase of the radius of the bubble when the radius has the values (*a*) 1 inch and (*b*) 4 inches. Ans. (*a*) 0.795 inch per second, (*b*) 0.050 inch per second.

7. A man 6 feet tall walks at a speed of 4 miles per hour under a lamp which is 10 feet from the ground. Find how fast ~~the~~ tip ~~of~~ the man's shadow lengthens ~~travels~~ when the horizontal distance from the lamp to the man is 20 feet. Ans. 6 miles per hour.

8. The two sides of a right angled triangle are 40 inches and x, and the hypothenuse is y. The side x is growing at the rate of 5 inches per second. How fast is y growing when $x = 30$ inches? Ans. 3 inches per second.

9. The two sides of a right angled triangle are x and y, and the hypothenuse is 50 inches. The side x is growing at the rate of 4 inches per second. How fast is y growing (*a*) when $x = 30$ inches, and (*b*) when $x = 40$ inches. Ans. (*a*) 3 inches per second. (*b*) 5.33 inches per second.

10. The observed temperatures of a vessel of cooling water after 1 minute, after 2 minutes, and so forth are as follows:

Elapsed Time in Minutes, t.	Observed Temperatures, T.
0	92.0
1	85.3
2	79.5
3	74.5
5	67.0
7	60.5
10	53.5
15	45.0
20	39.5

Plot these values of t and T, using the best grade of squared paper, draw a smooth curve through the plotted points, draw tangents to the curve at the points corresponding to (*a*) $T = 80°$, (*b*) $T = 65°$, and (*c*) $T = 50°$, and determine the rate of decrease of the temperature at each point by measuring the intercepts of these tangents on the axes of reference.

11. Find for what values of x the function $y = 6x^2 - 12x$ is an increasing function and for what values of x it is a decreasing function. Ans. y is an increasing function when x is greater than 1, and a decreasing function when x is less than 1.

12. Plot the values of x and y, where $y = x^3$, using the best grade of squared paper, draw a smooth curve through the plotted points, draw tangents to the curve at the points corresponding to $x = 1$, $x = 2$ and $x = 3$, and determine the corresponding values of $\frac{dy}{dx}$ by measuring the intercepts of the tangents on the axes of reference. Compare the values of $\frac{dy}{dx}$ thus found with the values as calculated from the formula $\frac{dy}{dx} = 3x^2$.

13. Plot the values of x and y where $y = \log_{10} x$. Take the values of $\log_{10} x$ from an ordinary table of logarithms and use the following values of x: 2, 3, 4, 5, 6, 7, 8, 9 and 10. Draw a smooth curve through the plotted points, draw tangents to the curve at the points corresponding to $x = 4$, $x = 6$ and $x = 8$, and determine the corresponding values of $\frac{dy}{dx}$ by measuring the intercepts of these tangents on the axes of reference. Compare the values of $\frac{dy}{dx}$ thus found with the values as calculated by the formula $\frac{dy}{dx} = \frac{\log_{10} e}{x} = \frac{0.4343}{x}$.

20. Determination of limiting relation between Δy and Δx by consideration of infinitesimals.—The value of $\frac{\Delta y}{\Delta x}$ is found in Arts. 10 and 11 as the sum of *a finite quantity* and *a quantity which approaches zero as* Δx *approaches zero.* Thus when $y = ax^3$ we found that $\frac{\Delta y}{\Delta x} = 3ax^2 + 3ax \cdot \Delta x + a(\Delta x)^2$, in which $3ax^2$ is a finite quantity and $3ax \cdot \Delta x + a(\Delta x)^2$ is a quantity which approaches zero as Δx approaches zero. In such a case it is very

easy to see what the limiting value of $\dfrac{\Delta y}{\Delta x}$ must be.

In many cases, however, it is desirable to find the limiting relation between Δy and Δx when Δx approaches zero *without deriving an expression for* $\dfrac{\Delta y}{\Delta x}$. For example let $y = ax^3$. Then from Art. 11 we have:

$$\Delta y = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{1}$$

Both members of this equation approach zero when Δx approaches zero, and it would therefore seem rather difficult to determine the limiting relation between Δy and Δx (when Δx approaches zero) from this equation as it stands. But when Δx is made as small as you please, then $(\Delta x)^2$ is infinitely smaller than Δx, and $(\Delta x)^3$ is infinitely smaller than $(\Delta x)^2$. For example let Δx be a millionth of a unit, then $(\Delta x)^2$ is a million-millionth of a unit, and $(\Delta x)^3$ is a million-million-millionth of a unit. Therefore the terms $3ax \cdot (\Delta x)^2$ and $a(\Delta x)^3$ become more and more nearly negligible in comparison with $3ax^2 \cdot \Delta x$ as Δx grows smaller and smaller in equation (1). The *limiting relation* between Δy and Δx may be found by writing dx and dy for Δx and Δy to indicate that we have proceeded to the limit, and by dropping every term which contains the square or any higher power of Δx (or the square or any higher power of Δy). This gives

$$dy = 3ax^2 \cdot dx \tag{2}$$

In this equation dy and dx are as small as you please and they are called *infinitesimals;* but $(dy)^2$ and $(dx)^2$ being infinitely smaller than dy and dx are called *infinitesimals of the second order*, $(dy)^3$ and $(dx)^3$ being infinitely smaller than $(dy)^2$ and $(dx)^2$ are called *infinitesimals of the third order*, and so on. *In any differential expression the lowest order infinitesimals, only, are significant; all terms containing higher order infinitesimals as factors may be dropped.*

Example 1.—Let it be required to differentiate the expression

$$y^2 = ax^3 \tag{3}$$

that is to say, let it be required to determine the relation between Δy and Δx when Δx is as small as you please. Writing $(y + \Delta y)$ for y and writing $(x + \Delta x)$ for x, we have

$$y^2 + 2y \cdot \Delta y + (\Delta y)^2 = ax^3 + 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{4}$$

whence, subtracting equation (3) from equation (4) member by member, we have

$$2y \cdot \Delta y + (\Delta y)^2 = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{5}$$

from which the desired limiting relation is found by dropping second and third order infinitesimals, so that we have:

$$2y \cdot dy = 3ax^2 \cdot dx \tag{6}$$

The value of y from equation (3) may be substituted in equation (6), giving:

$$2\sqrt{ax^3} \cdot dy = 3ax^2 \cdot dx \tag{7}$$

which can be simplified by cancellation, giving:

$$dy = \tfrac{3}{2}\sqrt{ax} \cdot dx \tag{8}$$

This dropping of higher order infinitesimals from differential expressions does not lead to merely approximate results, because in every case it is the limiting relation between Δy and Δx which is to be determined, that is, the relation when Δy and Δx are both as small as you please. Equation (8), for example, is rigorously true.

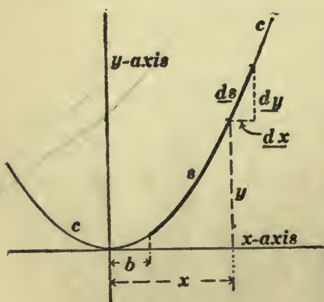


Fig. 13.

Example 2. Differential of the arc of a parabola.—The length s of the heavy portion of the curve cc,

Fig. 13, from $x = b$ to $x = x$ is a function of x, and from the infinitesimal triangle whose sides are dx, dy and ds we have

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{9}$$

or

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx \tag{10}$$

For example suppose the curve cc, Fig. 13, to be a parabola whose equation is:

$$y = kx^2 \tag{11}$$

then

$$\frac{dy}{dx} = 2kx \tag{12}$$

which, substituted in equation 10, gives:

$$ds = \sqrt{1 + 4k^2x^2} \cdot dx \tag{13}$$

21. Functions which have the same derivative.—The curve A, Fig. 14, defines y' as a function of x, and the curve B defines y

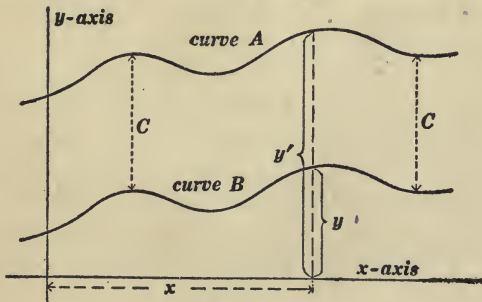


Fig. 14.

as a function of x. The steepness or grade of curve A is everywhere the same as the steepness or grade of curve B, that is, the derivative $\frac{dy'}{dx}$ is equal to the derivative $\frac{dy}{dx}$ for each value of x,

or expressed as an equation we have

$$\frac{dy'}{dx} = \frac{dy}{dx} \tag{1}$$

From the figure it is evident that the difference $y' - y$ is everywhere equal to the constant quantity C. Consequently when equation (1) is true we have

$$y' - y = \text{a constant} \tag{2}$$

That is to say, *two functions whose derivatives are equal have a constant difference.*

Examples.—When two men save money at the same rate there is a constant difference between the amounts they have saved; or, if they start even, their savings are equal. When two trains travel at the same speed they remain at a constant distance apart. The two functions ax^2 and $ax^2 + b$ have the same derivative with respect to x, b being a constant.

22. Example showing the use of calculus. Work required to stretch a spring.—Let W be the work required to stretch a spring from condition A to condition B, Fig. 2. It is evident that W is a function of e, and it is desired to find an algebraic expression for this function; that is, it is desired to find an equation expressing W in terms of e. *To do this the first step is to find an expression for the derivative* $\frac{dW}{de}$ by considering the amount of work ΔW that must be done to produce a slight additional elongation Δe.

The force F in Fig. 2 is, according to Hooke's law, proportional to e; that is,

$$F = ke \tag{1}$$

where k is a constant.

Imagine the force F to be increased by the amount ΔF so that the elongation would be increased by the amount Δe. Then the work done would be *greater than* $F \cdot \Delta e$ and *less than* $(F + \Delta F) \cdot \Delta e$ because the *average value* of the force which acts

on the spring while the added elongation Δe is being produced is greater than F and less than $F + \Delta F$. Therefore

$$\Delta W \text{ is greater than } F \cdot \Delta e \tag{2}$$

and

$$\Delta W \text{ is less than } (F + \Delta F) \cdot \Delta e \tag{3}$$

whence, dividing by Δe, we have:

$$\frac{\Delta W}{\Delta e} \text{ is greater than } F \text{ and less than } F + \Delta F \tag{4}$$

Therefore $\dfrac{\Delta W}{\Delta e}$ approaches F as its limit as ΔF (and also Δe) approaches zero. That is,

$$\frac{dW}{de} = F \tag{5}$$

or, using the value of F from equation (1), we have

$$\frac{dW}{de} = ke \tag{6}$$

Now in Art. 10 it is shown that ax^2 is a function whose derivative is $2ax$. Therefore:

$$ae^2 \text{ is a function whose derivative is } 2ae \tag{7}$$

Therefore, substituting $k = 2a$ (or $a = \frac{1}{2}k$) in (7) we have:

$$\tfrac{1}{2}ke^2 \text{ is a function whose derivative is } ke \tag{8}$$

The work W is also a function whose derivative is ke, according to equation (6). Therefore, according to Art. 21, we must have

$$W = \tfrac{1}{2}ke^2 + C \tag{9}$$

where C is a constant. To determine the value of the constant C we must *know* the value of W for some particular value of e, and of course we know that $W = 0$ when $e = 0$, that is, no work is required when the spring is not stretched at all. Therefore,

substituting this pair of values of W and e in equation (9), we have

$$0 = C \tag{10}$$

Substituting this value of C in equation (9) we have

$$W = \tfrac{1}{2}ke^2 \tag{11}$$

which is the desired expression for the work W.

23. Another example showing the use of calculus. The area under a parabola.—The curve cc, Fig. 15, represents a parabola of which the equation is:

$$y = px^2 \tag{1}$$

and it is required to find an expression for the area A.

Imagine the value of x in Fig. 15 to increase by the amount Δx,

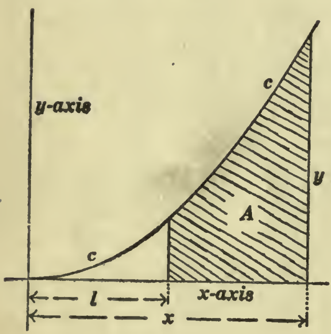


Fig 15.

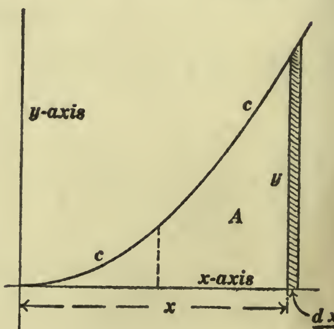


Fig. 16.

then the increment of A is the area of the narrow strip in Fig. 16. The width of this strip is Δx and its height is y ($= px^2$). Therefore

$$\Delta A = px^2 \cdot \Delta x \tag{2}$$

whence

$$\frac{\Delta A}{\Delta x} = px^2 \tag{3}$$

Now the width Δx in Fig. 16 must be infinitely small in order that the height of the strip may be considered to be equal to

y $(= px^2)$. That is, *we have already proceeded to the limit in writing down equation* (2), and it is evident that equation (3) gives the limiting value of $\frac{\Delta A}{\Delta x}$. Therefore we have

$$\frac{dA}{dx} = px^2 \tag{4}$$

In Art. 11 it was shown that:

$$ax^3 \text{ is a function whose derivative is } 3ax^2 \tag{5}$$

Therefore substituting $\frac{p}{3}$ for a we have:

$$\tfrac{1}{3}px^3 \text{ is a function whose derivative is } px^2 \tag{6}$$

The area A is also a function whose derivative with respect to x is px^2 according to equation (4), and therefore, according to Art. 21, we must have:

$$A = \tfrac{1}{3}px^3 + C \tag{7}$$

where C is a constant. To determine the value of the constant C we must *know* the value of A for some particular value of x. An inspection of Fig. 15 shows that $A = 0$ when $x = l$. Therefore, substituting this pair of values of A and x in equation (7) we have:

$$0 = \tfrac{1}{3}pl^3 + C \tag{8}$$

so that

$$C = -\tfrac{1}{3}pl^3 \tag{9}$$

Substituting this value of C in equation (7) we have:

$$A = \tfrac{1}{3}px^3 - \tfrac{1}{3}pl^3 \tag{10}$$

which is the desired expression for the area A in Fig. 15.

It is to be noted that x is a variable in the above discussion because we have arbitrarily made it vary in the derivation of equation (4), whereas p and l are both constant.

PROBLEMS.

In each of the first six following problems, find the rate of change of y with respect to x.

1. $y^2 = 3x$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{2}\sqrt{\dfrac{3}{x}}$.

2. $y^2 = ax^3 - 3x$, $\qquad \dfrac{dy}{dx} = \dfrac{3(ax^2 - 1)}{2\sqrt{ax^3 - 3x}}$.

3. $y^3 = 2ax^2 + 3x$, $\qquad \dfrac{dy}{dx} = \dfrac{4ax + 3}{3(2ax^2 + 3x)^{\frac{2}{3}}}$.

4. $y^2 = a^2 - x^2$, $\qquad \dfrac{dy}{dx} = \dfrac{-x}{\sqrt{a^2 - x^2}}$.

5. $y^2 = \dfrac{1}{x}$, $\qquad \dfrac{dy}{dx} = -\dfrac{1}{2x^{\frac{3}{2}}}$.

6. $y^3 = \dfrac{1}{x}$, $\qquad \dfrac{dy}{dx} = -\dfrac{1}{3x^{\frac{4}{3}}}$.

7. The volume V of a cone of which the half-angle at the apex is 30° is a function of the altitude x of the cone. Set up the derivative $\dfrac{dV}{dx}$. Ans. $\dfrac{dV}{dx} = \frac{1}{3}\pi x^2$.

8. Set up the derivative of the area A of the curved surface of the cone of problem 7 with respect to x. Ans. $\dfrac{dA}{dx} = \frac{4}{3}\pi x$.

9. The volume V of a cone of which the half-angle at the apex is 30° is a function of the slant height x of the cone. Set up the derivative $\dfrac{dV}{dx}$. Ans. $\dfrac{dV}{dx} = \dfrac{\sqrt{3}}{8}\pi x^2$.

10. The kinetic energy W of a disk of steel rotating at n revolutions per second is a function of the radius r of the disk. Set up the derivative $\dfrac{dW}{dr}$, the thickness of the disk being 10 centimeters. Ans. $\dfrac{dW}{dr} = 78\pi n^2 r^3$

Note.—The density of steel is 7.8 grams per cubic centimeter and the kinetic energy of a particle in ergs is $\frac{1}{2}mv^2$ where m is the mass of the particle in grams and v is the velocity of the particle in centimeters per second.

11. The shaded area in Fig. $p11$ is a function of the angle θ. Find the derivative of the area with respect to θ, the angle θ being expressed in radians and the equation of the curve cc being $r = k\theta + b$, where k and b are constants.

Ans. $\dfrac{dA}{d\theta} = \frac{1}{2}(k\theta + b)^2$.

12. The length of the arc s in Fig. $p12$ is a function of the

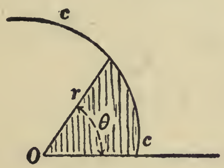


FIG. $p11$.

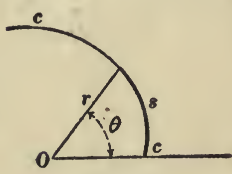


FIG. $p12$.

angle θ. Find the derivative of s with respect to θ, the equation of the curve cc being as given in problem 11. Ans. $\dfrac{ds}{d\theta} = \sqrt{k^2 + r^2}$.

24. The differential equation. Law of growth of a function.—Let y be an undetermined function of x concerning which it is known that y increases $2ax$ times as fast as x. This property of the unknown function may be expressed by the equation:

$$\frac{dy}{dx} = 2ax \tag{1}$$

This equation expresses the law of growth of the function y; the independent variable x being assumed to grow steadily in value. Any equation which expresses the law of growth of a function is called a *differential equation*.

Examples. (a) A man A saves money at a constant rate.* This statement is a verbal expression of a differential equation.

* Money saved is here assumed to be a continuous variable. See Art. 5.

To reduce the statement to algebra, let y be the amount of money saved by A, then the above statement is equivalent to

$$\frac{dy}{dt} = k \tag{2}$$

where k is a constant.

(*b*) One man A saves money twice as fast as another man B. This statement also is a verbal expression of a differential equation. To reduce it to algebra let y be the amount of money saved by A and let x be the amount of money saved by B. Then while B saves Δx dollars A must save twice as much according to the above statement. Therefore $\Delta y = 2\Delta x$ or $\frac{\Delta y}{\Delta x} = 2$, or

$$\frac{dy}{dx} = 2 \tag{3}$$

(*c*) Equation (6), Art. 22, is a differential equation, and equation (4), Art. 23, is a differential equation.

25. Differentiation and integration.—The finding of the derivative of a function as exemplified in Arts. 10 and 11 is called *differentiation.*

Let y be an unknown function of x which is known to increase $2ax$ times as fast as x. That is to say, it is known concerning y that

$$\frac{dy}{dx} = 2ax \tag{1}$$

Now when one has previously found that ax^2 *is a function of* x *whose derivative is equal to* $2ax$ *one is able to recognize that the unknown function* y *whose law of growth is given by equation* (1) *must be equal to* $ax^2 + C$, *where* C *is a constant,* as explained in Art. 21. This recognition of a function from its derivative is called *integration.*

Symbol of integration.—Let y stand for $ax^2 + C$, then

$$\frac{dy}{dx} = 2ax \tag{2a}$$

that is to say,

$$\textit{the derivative of } (ax^2 + C) \textit{ is equal to } 2ax \tag{2b}$$

Or, using the differential notation which is explained in Art. 19, we have

$$dy = 2ax \cdot dx \tag{2c}$$

that is to say,

$$\textit{the differential of } (ax^2 + C) \textit{ is equal to } 2ax \cdot dx \tag{2d}$$

All of these equations (2*a*), (2*b*), (2*c*) and (2*d*) express the same thing or the same relation, and this relation may be expressed in an inverse manner by saying:

$$(ax^2 + C) \textit{ is the integral of } 2ax \cdot dx \tag{3a}$$

or, in symbolic notation:

$$ax^2 + C = \int 2ax \cdot dx \tag{3b}$$

The symbol $\int$ means *integral of*. The undetermined constant C is called the *constant of integration*.

A differential equation does not completely determine the function whose law of growth it expresses.—This is evident when we consider that the function $y + C$ has the same derivative (the same law of growth) whatever the value of the constant C may be.

Thus the amount of a man's savings cannot be calculated when his rate of saving *alone* is known; one must know also how much money the man had on a given date. Thus a man who had $1,000 on January 1, 1912, and who saves $10 per month would have $1,000 + 10m$ dollars at any time m months after January 1. The amount of a man's savings is completely determined as a function of elapsed time by knowing (*a*) his rate of saving and (*b*) how much he had on a given date.

In general a function y is completely determined* by knowing

* This statement will have to be modified when we come to consider differential equations of higher orders, and when we come to consider partial differential equations.

(*a*) its law of growth and (*b*) its value corresponding to any given value of the independent variable x or t as the case may be. This matter is fully illustrated in Arts. 22 and 23.

26. Indefinite integrals and definite integrals.—Let y be a function of x concerning which nothing is known except that $\frac{dy}{dx}$ is equal, say, to $2ax$; or, using differential notation, we know that

$$dy = 2ax \cdot dx \tag{1}$$

Then according to Arts. 10 and 21 we know that y must be given by the equation:

$$y = ax^2 + C \tag{2}$$

where C is a constant which may have any value whatever. This equation (2) expresses what is called the *indefinite integral* of equation (1).

Let it be required to find the growth of y *while* x *grows from any given value* l *to any other given value* L. This growth of y is called a *definite integral* of equation (1), and the two given values of x are called the *boundaries* or *limits** of the definite integral.

Now, according to equation (2), we have $y = al^2 + C$ when $x = l$, and $y = aL^2 + C$ when $x = L$; and the difference between these two values of y, namely, $aL^2 - al^2$, is the desired growth of y from $x = l$ to $x = L$. That is, the definite integral of equation (1) between the boundaries or limits $x = l$ and $x = L$ is equal to $aL^2 - al^2$.

Examples of definite integrals.—Consider the area A under the parabola in Fig. 15. The differential of this area is

$$dA = px^2 \cdot dx \tag{3}$$

and the value of the area is

$$A = \tfrac{1}{3}px^3 - \tfrac{1}{3}pl^3 \tag{4}$$

* The word *limit* as here used has a very different meaning from the word *limit* as used in Arts. 4 and 7.

as explained in Art. 23. Now equation (4) is the definite integral of equation (3) from $x = l$ to $x = x$ (meaning any value of x whatever).

Symbolic representation of a definite integral.—The usual method of expressing a definite integral is

$$A = \int_l^x px^2 \cdot dx \tag{5}$$

where A is the area under the parabola in Fig. 15, the differential of A being given by equation (3) above.

Similarly, the work W required to stretch the spring in Fig. 2 is found as a definite integral in Art. 22, and to express W symbolically as a definite integral we write

$$W = \int^e ke \cdot de \tag{6}$$

The value of this definite integral is $\frac{1}{2}ke^2$ as explained in Art. 22.

27. A definite integral interpreted as the limit of a sum.—Imagine the area A of Fig. 15 to be divided into narrow strips as shown in Fig. 17, the width of successive strips being represented by successive increments Δx of x, starting at $x = l$ and extending to any given value of x. This given value of x is represented by L in Fig. 17, because it is now desirable to think of x as having various intermediate values (between $x = l$ and $x = L$).

The width of any strip is Δx and its altitude is px^2, where x is the distance of the particular strip from the y-axis. Therefore the area of the strip is $px^2 \cdot \Delta x$ and *the sum of the areas of all the strips is more and more nearly equal to the area A of Fig. 15 as the number of the strips is made greater and greater.*

The area A is equal to $\int_l^L px^2 \cdot dx$ as explained in Art. 26, and the sum of all the strips in Fig. 17 may be represented symbolically as $\sum_l^L px^2 \cdot \Delta x$. When the number of strips is made greater and greater (successive increments Δx made smaller and smaller) then $\sum_l^L px^2 \cdot \Delta x$ approaches $\int_l^L px^2 \cdot dx$ as a limit. That is, the limit of $\sum_l^L px^2 \cdot \Delta x$ is $\int_l^L px^2 \cdot dx$.

28. Plan of this treatise.—The application of calculus involves three important things. Thus in Art. 22 the following three things are involved: (*a*) The finding of the derivative $\frac{dW}{de}$ by making use of fundamental principles in physics, (*b*) the becoming familiar with the derivative of ax^2 in Art. 10, and (*c*) the substitution of $\frac{1}{2}k$ for a and e for x so as to get a known function whose derivative is ke.

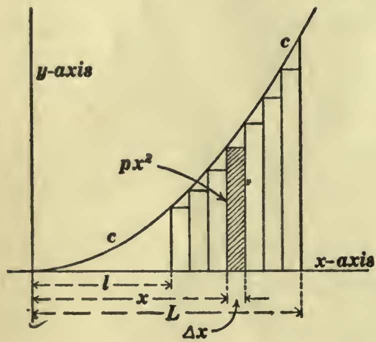


Fig. 17.

The application of calculus in engineering involves these three steps in nearly every case, and the following development of calculus consists of three parts (more or less intermingled) as follows:

(*a*) The setting up of derivative expressions (differential equations) with the help of fundamental principles in physics.

(*b*) The study of derivatives of algebraic functions; and

(*c*) Practice in transforming derivative expressions to standard forms for purposes of recognition.

The most important branches of calculus which do not fall into this outline are the discussion of maximum and minimum values of functions and the expansion of functions in series

A table of functions and their derivatives is given in Appendix B. The student should get into the habit of referring to this table.*

PROBLEMS.

1. A force of 100 pounds produces 4 inches of elongation of a spring. Find the value of k in equation (1) of Article 22, and express the unit in terms of which k is found. Ans. 25 pounds per inch.

* Or of differential expressions and their integrals.

2. Using the spring referred to in problem 1, find the work required to produce (*a*) an elongation of 2 inches, and (*b*) an elongation of 4 inches. Ans. (*a*) 50 pound-inches, (*b*) 200 pound-inches.

3. Using the spring referred to in problem 1, find how much work is required to increase the elongation from 3 inches to 4 inches. Ans. 87.5 pound-inches.

4. Consider the parabola $y = 2x^2$. Find the area under this curve between ordinates at $x = 2$ and $x = 10$. Ans. 661.3.

5. Find an expression for the area under the curve $y = qx$ between an ordinate at $x = a$ and the ordinate corresponding to any abscissa x. Ans. $A = \frac{1}{2}qx^2 - \frac{1}{2}qa^2$.

CHAPTER II.

FORMULAS FOR DIFFERENTIATION AND INTEGRATION.

29. Differentiation by development and differentiation by rule.—Throughout the previous chapter the derivative of a function (as in Arts. 10 and 11) or the differential of a function (as in Art. 20) is determined by developing an expression for $\frac{\Delta y}{\Delta x}$ or for Δy, and considering the limiting form of this expression as Δy and Δx approach zero. This is the fundamental method of differentiation. *It is possible, however, by means of this fundamental method to establish a few rules which enable one to write down the derivative or differential of almost any algebraic function.* It is the object of this chapter to establish these rules and give examples of their use.

Differentiation may be defined in general as the finding of the rate of change of a function. Thus problem 10 on page 21 illustrates what may be called *graphical differentiation.* And to determine a rate by actually measuring the change which takes place in a given time may be called *physical differentiation.* Thus the speed of a runner is determined by measuring the distance traveled by the runner in a given time, or rather by measuring the time required for the runner to travel a given distance. To determine the rate of supply of water by a small spring is to "differentiate the spring" and the simplest method is to make the differentiation by means of a dipper and a dollar watch.

30. Differentiation of $\mathbf{ax^n}$.—Let

$$\mathbf{y = ax^n} \tag{1}$$

Then

$$\mathbf{dy = nax^{n-1} \cdot dx} \tag{2}$$

for any value of n. We shall here prove equation (2) when n is a positive integer.*

* A general proof is given in Art. 39.

Proof.—Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (1) we have:

$$y + \Delta y = a(x + \Delta x)^n \tag{3}$$

When n is a positive integer $(x + \Delta x)^n$ can be expanded by the binomial theorem,* and we have

$$y + \Delta y = ax^n + nax^{n-1} \cdot \Delta x + \frac{n(n-1)ax^{n-2}(\Delta x)^2}{1 \cdot 2} + \text{etc.} \tag{4}$$

Subtracting equation (1) from equation (4), member by member, we have:

$$\Delta y = nax^{n-1} \cdot \Delta x + \frac{n(n-1)ax^{n-2}(\Delta x)^2}{1 \cdot 2} + \text{etc.} \tag{5}$$

Whence, writing dy for Δy and dx for Δx, and dropping infinitesimals of second and higher orders, we have:

$$dy = nax^{n-1} \cdot dx \tag{2}$$

31. Differentiation of the sum of two or more functions.—Let u and v be any two functions of x, and let

$$\mathbf{y} = \mathbf{u} + \mathbf{v} \tag{1}$$

Then

$$\mathbf{dy} = \mathbf{du} + \mathbf{dv} \tag{2}$$

or, using derivative notation,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{du}}{\mathbf{dx}} + \frac{\mathbf{dv}}{\mathbf{dx}} \tag{3}$$

Proof.—It is evident that the increment of y is equal to the sum of the increments of u and v, that is, $\Delta y = \Delta u + \Delta v$; and this is true however small the increments may be. Therefore when the increments are infinitesimal we have $dy = du + dv$.

* The binomial theorem is:

$$(a + b)^n = a^n + \frac{na^{n-1}b}{1} + \frac{n(n-1)a^{n-2}b^2}{1 \cdot 2} + \frac{n(n-1)(n-2)a^{n-3}b^3}{1 \cdot 2 \cdot 3} + \text{etc.}$$

When n is not an integer this becomes an infinite series, and an infinite series cannot be used in a mathematical argument unless the question of its convergence is carefully considered. See Art. 35.

Also it is evident that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

and this is true however small the increments may be. Therefore when the increments are infinitesimal we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Explanation.—The fact which is expressed by equation (3) is extremely simple and it is self-evident when it is expressed in familiar terms as follows: The amount of money earned by one man is u and his rate of earning is 2 dollars per day $\left(= \frac{du}{dt}\right)$; the amount of money earned by another man is v and his rate of earning is 3 dollars per day $\left(= \frac{dv}{dt}\right)$. The money earned by both men together is $u + v$ and their combined rate of earning is 5 dollars per day, which is the rate of change of $(u + v)$.

32. The differentiation of the product of two functions.—Let u and v be any two functions of x, and let

$$\mathbf{y = uv} \tag{1}$$

Then

$$\mathbf{dy = u \cdot dv + v \cdot du} \tag{2}$$

Proof.—Writing $y + \Delta y$ for y,
$u + \Delta u$ for u and
$v + \Delta v$ for v in equation (1) we have:

$$y + \Delta y = (u + \Delta u)(v + \Delta v) \tag{3}$$

or

$$y + \Delta y = uv + u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v \tag{4}$$

Whence, subtracting equation (1) from equation (3) member by member, we have

$$\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v \tag{5}$$

But $\Delta u \cdot \Delta v$ is a second order infinitesimal, and therefore when

Δu and Δv approach zero we may drop $\Delta u \cdot \Delta v$ and we have the limiting relation:

$$dy = u \cdot dv + v \cdot du$$

Example.—Let $u = ax^2 + bx$, let $v = cx^3 + 1$ and let $y = (ax^2 + bx)(cx^3 + 1)$. To differentiate this expression substitute the values of u and v, and the values of

$$du\ (= 2ax \cdot dx + b \cdot dx) \quad \text{and} \quad dv\ (= 3cx^2 \cdot dx)$$

in equation (2), and we have

$$dy = (ax^2 + bx) \times 3cx^2 \cdot dx + (cx^3 + 1)(2ax + b) \cdot dx$$

33. Differentiation of the quotient of two functions.—Let u and v be any two functions of x, and let

$$\mathbf{y} = \frac{\mathbf{u}}{\mathbf{v}} \tag{1}$$

Then

$$\mathbf{dy} = \frac{\mathbf{v} \cdot \mathbf{du} - \mathbf{u} \cdot \mathbf{dv}}{\mathbf{v}^2} \tag{2}$$

Proof.—From equation (1) we have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v} \tag{3}$$

whence, subtracting equation (1) from equation (3) member by member, we have

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \tag{4}$$

Reducing $\frac{u + \Delta u}{v + \Delta v}$ and $\frac{u}{v}$ to a common denominator, equation (4) becomes:

$$\Delta y = \frac{v \cdot \Delta u - u \cdot \Delta v}{v^2 + v \cdot \Delta v} \tag{5}$$

But as Δu and Δv approach zero the denominator approaches v^2 as its limit. Therefore equation (5) reduces to equation (2).

Example.—Let it be required to differentiate the function:

$$y = \frac{ax^2 + bx}{cx^3 + d} \tag{6}$$

Let $u = ax^2 + bx$ and let $v = cx^3 + d$. Substitute these values of u and v, and the values of $du(= 2ax \cdot dx + b \cdot dx)$ and $dv(= 3cx^2 \cdot dx)$ in equation (2) and we get the desired result.

34. Differentiation of a function of a function.—*Let* y *be a function of* x *and let* z *be a function of* y. Then y changes $\frac{dy}{dx}$ times as fast as x, and z changes $\frac{dz}{dy}$ times as fast as y. Therefore it is evident that z changes $\frac{dz}{dy} \times \frac{dy}{dx}$ times as fast as x. That is,

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} \tag{1}$$

The symbols $\frac{dy}{dx}$ and $\frac{dz}{dy}$ are so cumbersome that the proposition upon which this equation is based is not easy to understand when it is stated as above. Reduced to its simplest terms the proposition is as follows: If z changes 5 times as fast as y and if y changes 7 times as fast as x, then z changes 5×7 times as fast as x!

Example.—Let

$$z = a(x^2 + x + 1)^3$$

To differentiate this expression one can write y for $x^2 + x + 1$. Then $z = ay^3$ and $\frac{dz}{dy} = 3ay^2$. But $\frac{dy}{dx} = 2x + 1$. Therefore, using equation (1), we have $\frac{dz}{dx} = 3a(x^2 + x + 1)^2(2x + 1)$.

For problems see group 1 in the Appendix.

DIFFERENTIATION OF LOGARITHM.

35. Convergent series.—Before the derivative of *logarithm of* x can be found it is helpful to consider the two series of fractions:

	a	b	c	d	e	$\cdots$	n
1st series	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\cdots$	$\frac{1}{2^n}$
2d series	$\frac{1}{\lfloor 2}$	$\frac{1}{\lfloor 3}$	$\frac{1}{\lfloor 4}$	$\frac{1}{\lfloor 5}$	$\frac{1}{\lfloor 6}$	$\cdots$	$\frac{1}{\lfloor n}$

where $\lfloor n$ stands for the product of all positive integers from 1 to n inclusive, *factorial n*, as it is sometimes called.

The first series may be brought before the reader most distinctly by means of the following schedule:

	Fraction of original sum which is spent each day.
Half of a sum of money is spent the first day.........	$\frac{1}{2}$
Half of the remainder is spent the second day.......	$\frac{1}{4}$
Half of the remainder is spent the third day......	$\frac{1}{8}$
Half of the remainder is spent the fourth day...	$\frac{1}{16}$
etc., etc.,	etc.

If one were to follow this schedule indefinitely the original sum of money would never be entirely spent, because there is always an unspent remainder; but the remainder would grow smaller and smaller, and the amount spent would approach, *as nearly as you please,* to the entire original sum. This is equivalent to saying that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ etc. can never be equal to unity, but it can be made to approach unity, *as nearly as you please,* by adding together a larger and larger number of the successive fractions of the series. Such a series of fractions is called a *convergent series,* and the limit of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ etc., as a greater and greater number of the successive fractions are included, is called the *sum of the series.* The sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ etc. is unity.

A series need not be convergent merely because the successive fractions of the series grow smaller and smaller. Thus $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$ etc. grows

larger and larger *without limit* as a greater and greater number of the successive fractions are included. Such a series of fractions is called a *divergent series.*

To show that the series $\frac{1}{\underline{|2}}, \frac{1}{\underline{|3}}, \frac{1}{\underline{|4}}, \frac{1}{\underline{|5}}$, etc. is a convergent series it is sufficient to arrange the two series, as in the following table, showing how each fraction is related to the preceding fraction in each series:

	a	b	c	d	
1st series	$\frac{1}{2}$	$\frac{1}{2}$ of a	$\frac{1}{2}$ of b	$\frac{1}{2}$ of c	etc.
2d series	$\frac{1}{2}$	$\frac{1}{3}$ of a	$\frac{1}{4}$ of b	$\frac{1}{5}$ of c	etc.

Each fraction of the second series is equal to or smaller than the corresponding fraction of the first series. Therefore, if the first series is convergent, the second series must also be convergent.

The particular series which is used in finding the derivative of a logarithm is

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \text{etc.} \tag{1}$$

and this series is convergent because it is identical to the second series in the above table with the exception of the first two terms. The sum of this series, correct to seven decimal places, is:

$$e = 2.7182818 \tag{2}$$

36. The limit of $\left(1 + \frac{1}{z}\right)^z$ as z approaches infinity.—This limit is important in finding the derivative of a logarithm. Let us assume that z is always a positive integer. In this case $\left(1 + \frac{1}{z}\right)^z$ can be expanded by the binomial theorem,* giving:

$$\begin{aligned}\left(1 + \frac{1}{z}\right)^z = 1 + \frac{1}{\underline{|1}} z \left(\frac{1}{z}\right) + \frac{1}{\underline{|2}} z(z-1)\left(\frac{1}{z}\right)^2 \\ + \frac{1}{\underline{|3}} z(z-1)(z-2)\left(\frac{1}{z}\right)^3 + \text{etc.}\end{aligned} \tag{3}$$

* See foot-note to Art. 30. Substitute $a = 1$, $b = \frac{1}{z}$ and $n = z$. and we have equation (3).

$$=1+1+\frac{1}{\lfloor 2}\left(1-\frac{1}{z}\right)+\frac{1}{\lfloor 3}\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)+\text{etc.} \quad (4)$$

But as z approaches infinity $\left(1-\frac{1}{z}\right)$, $\left(1-\frac{2}{z}\right)$, etc., approach unity, and therefore the limiting value of $\left(1+\frac{1}{z}\right)^z$, as z approaches infinity, is found by writing unity for each of the expressions $\left(1-\frac{1}{z}\right)$, $\left(1-\frac{2}{z}\right)$, etc., in equation (4), and when this is done the second member of equation (4) reduces to $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+$ etc. Therefore the limiting value of $\left(1+\frac{1}{z}\right)^z$ as z approaches infinity is 2.7182818, or e, as explained in the previous article.

To prove that the limit of $\left(1+\frac{1}{z}\right)^z$ *is* e *when* z *approaches infinity, but when* z *is not always a positive integer.* Let s be any positive value whatever of z, and let m be a positive integer such that z lies between m and $m+1$. Then $\left(1+\frac{1}{z}\right)^z$ lies between $\left(1+\frac{1}{m}\right)^m$ and $\left(1+\frac{1}{m+1}\right)^{m+1}$. Now as m approaches infinity $\left(1+\frac{1}{m}\right)^m$ and $\left(1+\frac{1}{m+1}\right)^{m+1}$ both approach e as a limit; and therefore $\left(1+\frac{1}{z}\right)^z$, which is always between $\left(1+\frac{1}{m}\right)^m$ and $\left(1+\frac{1}{m+1}\right)^{m+1}$, also approaches e as a limit.

To show that $\left(1+\frac{1}{z}\right)^z$ approaches e as a limit when z approaches negative infinity. Let $z=-r$. Then $\left(1+\frac{1}{z}\right)^z=\left(1-\frac{1}{r}\right)^{-r}$ where z is negative and r is positive. Now

$$\left(1-\frac{1}{r}\right)^{-r}=\left(\frac{r-1}{r}\right)^{-r}=\left(\frac{r}{r-1}\right)^{r}=\left(1+\frac{1}{r-1}\right)^{r}$$
$$=\left(1+\frac{1}{r-1}\right)^{r-1}\cdot\left(1+\frac{1}{r-1}\right) \quad (5)$$

but $\left(1+\frac{1}{r-1}\right)$ approaches unity as r approaches infinity and $\left(1+\frac{1}{r-1}\right)^{r-1}$

approaches e as r approaches infinity. Therefore every member of equation (5) approaches e as r approaches infinity. Consequently $\left(1+\frac{1}{z}\right)^z$ approaches e when z approaches negative infinity.

PROBLEM.

One dollar placed at compound interest for 10 years at 6 per cent. amounts to $(1 + 0.06)^{10}$ dollars, when accrued interest is added to the principal once a year. If accrued interest is added to the principle n times per year, one dollar after 10 years would amount to $\left(1 + \frac{0.06}{n}\right)^{10n}$. Find what one dollar would amount to after 10 years at 6 per cent. compound interest (a) when accrued interest is added to principle once per year, that is, when $n = 1$, (b) when $n = 2$, and (c) when accrued interest bears interest without any delay whatever, that is, when $n = \infty$. Ans. (a) \$1.791, (b) \$1.803, (c) \$1.822.

Note.—$\left(1 + \frac{0.06}{n}\right)^{10n} = \left[\left(1 + \frac{1}{z}\right)^z\right]^a$ where $z = \frac{n}{0.06}$ and $a = 0.6$.

37. Differentiation of the logarithm of x.—Let y be the number of times that a given number a must be multiplied by itself to give x. Then $x = a^y$, and y is called *the logarithm of* x *to the base* a. The base of the common system of logarithms is 10, and the base of the Napierian system of logarithms is 2.7182818 or e. *Napierian logarithms are used throughout this treatise except where it is otherwise expressly stated.**

Let

$$\mathbf{y} = \log \mathbf{x} \tag{1}$$

Then

$$\mathbf{dy} = \frac{\mathbf{dx}}{\mathbf{x}} \tag{2}$$

Proof.—Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in

* Napierian $\log x = \frac{\log_{10} x}{\log_{10} e}$.

equation (1) we have:

$$y + \Delta y = \log(x + \Delta x) \tag{3}$$

whence, subtracting equation (1) from equation (3) member by member, we have:

$$\Delta y = \log(x + \Delta x) - \log x \tag{4}$$

or

$$\frac{\Delta y}{\Delta x} = \frac{\log(x + \Delta x) - \log x}{\Delta x} \tag{5}$$

But $\log(x + \Delta x) - \log x$ is equal to $\log\left(\frac{x + \Delta x}{x}\right)$ or to $\log\left(1 + \frac{\Delta x}{x}\right)$; and $\frac{1}{\Delta x}\log\left(1 + \frac{\Delta x}{x}\right)$ is equal to $\log\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}$.

Therefore equation (5) reduces to

$$\frac{\Delta y}{\Delta x} = \log\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} \tag{6}$$

and this may be written

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\log\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \tag{7}$$

or, writing z for $\frac{x}{\Delta x}$, we have:

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\log\left(1 + \frac{1}{z}\right)^{z} \tag{8}$$

But, as Δx approaches zero $z\left(=\frac{x}{\Delta x}\right)$ approaches infinity, and the limiting value of $\left(1 + \frac{1}{z}\right)^{z}$ as z approaches infinity is e, as explained in Art. 36. Therefore as z approaches infinity equation (8) becomes:

$$\frac{dy}{dx} = \frac{1}{x}\cdot\log e \tag{9}$$

But the Napierian logarithm of e is unity because e is equal to e^1.

Therefore equation (9) reduces to

$$\frac{dy}{dx} = \frac{1}{x} \quad \text{or} \quad dy = \frac{dx}{x}$$

Differentiation of $\log_a x$.—Let

$$y = \log_a x \tag{10}$$

This means that

$$x = a^y \tag{11}$$

Therefore, taking Napierian logarithm of each member of this equation, we have

$$\log x = y \log a$$

or

$$y = \frac{1}{\log a} \cdot \log x \tag{12}$$

where $\dfrac{1}{\log a}$ is, of course, a constant. Therefore, according to equation (1) and (2) we have

$$dy = \frac{1}{\log a} \cdot \frac{dx}{x} = \frac{\log_a e \cdot dx}{x} \tag{13}$$

38. Differentiation of the exponential function.—Let

$$y = Ae^{kx} \tag{1}$$

where A and k are constants and $e = 2.7182818$. Then:

$$\frac{dy}{dx} = kAe^{kx} = ky \tag{2}$$

Proof.—Taking logarithms of both members of equation (1) we have

$$\log y = \log A + kx \tag{3}$$

But the differential* of $\log y$ is $\dfrac{dy}{y}$ according to Art. 37. There-

* It is very important that the student keep in mind what a differential is. The differential of $\log y$ is the infinitesimal increment of $\log y$ due to an infinitesimal increment, dy, of y.

fore, by differentiating, both members of equation (3) we have:

$$\frac{dy}{y} = k \cdot dx \tag{5}$$

whence equation (3) is obtained by solving for $\frac{dy}{dx}$; and the value of $y (= Ae^{kx})$ may be substituted for y in the resulting equation.

Differentiation of a^{kx}.—Let

$$\mathbf{y} = \mathbf{a}^{kx} \tag{6}$$

where a and k are constants. Taking logarithm of each member of (6), we have

$$\log y = kx \log a \tag{7}$$

Differentiating we have

$$\frac{dy}{y} = k \log a \cdot dx \tag{8}$$

Therefore, solving for $\frac{dy}{dx}$ and substituting the value of y from equation (6), we have:

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{k} \log \mathbf{a} \cdot \mathbf{a}^{kx} = \mathbf{k} \log \mathbf{a} \cdot \mathbf{y} \tag{9}$$

PROBLEM.

Given the logarithm of 2 to base 10 find the approximate value of logarithm of 2.01. Ans. 0.3032.

Note.—Consider $\log_{10} x$. The derivative of $\log_{10} x$ multiplied by a small increment of x gives the approximate value of the increment of $\log_{10} x$.

39. General proof of equation (2) of Art. 30.—Let

$$y = ax^n \tag{1}$$

where n has any value, integral or fractional, positive or negative. Taking logarithms of both members of (1) we have:

$$\log y = \log a + n \log x \tag{2}$$

Differentiating, we have

$$\frac{dy}{y} = n\frac{dx}{x} \tag{3}$$

$\log a$ being a constant. Substitute ax^n for y in this equation and solve for dy, and we have

$$dy = nax^{n-1} \cdot dx \tag{4}$$

For problems see group 2 in the Appendix.

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS.

40. Limit of $\dfrac{\sin\phi}{\phi}$ as ϕ approaches zero.—It is a fundamental principle in geometry that if a polygon of n sides be inscribed in a circle,* *the sum of the sides of the polygon approaches the length of the circular arc as a limit when n approaches infinity.* The idea that a circular arc has a definite length depends upon this principle.

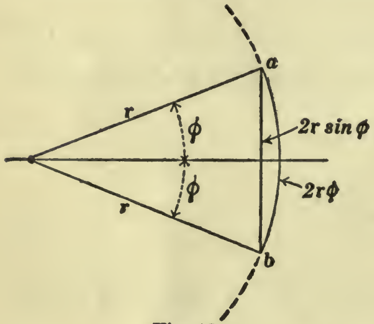


Fig. 18.

Consider Fig. 18. The arc ab is equal to $2r\phi$, ϕ being expressed in radians, and the chord ab is equal to $2r \sin\phi$. Therefore:

$$\frac{\text{chord } ab}{\text{arc } ab} = \frac{2r\sin\phi}{2r\phi} = \frac{\sin\phi}{\phi}.$$

But $\dfrac{\text{chord}}{\text{arc}}$ approaches unity as its limit when ϕ approaches zero; and therefore $\dfrac{\sin\phi}{\phi}$ must also approach unity as its limit when ϕ approaches zero.

41. Differentiation of $\sin x$.—Let

$$y = \sin x \tag{1}$$

Then

$$dy = \cos x \cdot dx \tag{2}$$

Proof.—Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x, we

* Or any continuous curve.

have:

$$y + \Delta y = \sin(x + \Delta x) \tag{3}$$

Whence, subtracting equation (1) from equation (3) member by member and solving for $\frac{\Delta y}{\Delta x}$, we have

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \tag{4}$$

Using the trigonometric formula:

$$\sin a - \sin b = 2 \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2}(a + b) \tag{5}$$

equation (3) may be reduced to:

$$\frac{\Delta y}{\Delta x} = \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} \cdot \cos(x + \tfrac{1}{2}\Delta x) \tag{6}$$

but as Δx approaches zero $\cos(x + \frac{1}{2}\Delta x)$ approaches $\cos x$ as its limit, and $\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}$ approaches unity as its limit. Therefore, from equation (6) we have:

$$\frac{dy}{dx} = \cos x \quad \text{or} \quad dy = \cos x \cdot dx$$

42. Differentiation of cos x.—Let

$$\mathbf{y = \cos x} \tag{1}$$

Then

$$\mathbf{dy = -\sin x \cdot dx} \tag{2}$$

Proof.—Proceeding as in the previous article we get

$$\frac{\Delta y}{\Delta x} = \frac{\cos(x + \Delta x) - \cos x}{\Delta x} \tag{3}$$

and using the trigonometric formula:

$$\cos a - \cos b = -2 \sin \tfrac{1}{2}(a - b) \sin \tfrac{1}{2}(a + b) \tag{4}$$

we obtain

$$\frac{\Delta y}{\Delta x} = -\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} \cdot \sin(x + \tfrac{1}{2}\Delta x) \tag{5}$$

and from this we obtain

$$\frac{dy}{dx} = -\sin x \quad \text{or} \quad dy = -\sin x \cdot dx$$

For problems see group 3 in the Appendix.

EXAMPLES SHOWING USE OF FORMULAS FOR DIFFERENTIATION.

43. Differentiation of tan x.—In order to differentiate $\tan x$ the familiar formula $\tan x = \dfrac{\sin x}{\cos x}$ may be used as follows:

$$y = \tan x = \frac{\sin x}{\cos x} \tag{1}$$

Proceeding according to Art. 33, let $u = \sin x$ so that $du = \cos x \cdot dx$, and let $v = \cos x$ so that $dv = -\sin x \cdot dx$. Then using equation (2) of Art. 33, we have:

$$dy = \frac{\cos^2 x \cdot dx + \sin^2 x \cdot dx}{\cos^2 x} = \frac{dx}{\cos^2 x} \tag{2}$$

Any trigonometric function can be differentiated by expressing the function in terms of sine and cosine and using the formulas of Arts. 41 and 42.

As a further example, consider

$$i = I \sin \omega t \tag{3}$$

in which I and ω are constants and t is elasped time reckoned from any chosen instant, and let it be required to find the rate of change of i, namely, $\dfrac{di}{dt}$. Now ωt is a function of t and $\sin \omega t$ is a function of ωt. Therefore we may use the formula for the differentiation of a function of a function, as explained in Art. 34. Let

$$z = \omega t \tag{4}$$

then

$$dz = \omega \cdot dt \tag{5}$$

Writing z for ωt in equation (3) we have:

$$i = I \sin z \tag{6}$$

which, by differentiation gives:

$$di = I \cos z \cdot dz \tag{7}$$

Whence, substituting ωt for z and $\omega \cdot dt$ for dz we have

$$di = \omega I \cos \omega t \cdot dt \tag{8}$$

or

$$\frac{di}{dt} = \omega I \cos \omega t \tag{9}$$

For problems see group 4 in the Appendix.

44. Differentiation of $\sin^{-1} x$.—The inverse trigonometric functions can all be differentiated with the help of the formulas already established. For example let

$$\mathbf{y = \sin^{-1} x} \tag{1}$$

this means that

$$\sin y = x \tag{2}$$

which, by differentiation, gives:

$$\cos y \cdot dy = dx \tag{3}$$

but $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$, according to equation (2). Therefore, substituting this value for $\cos y$ in equation (3) and solving for dy, we have

$$\mathbf{dy = \frac{dx}{\sqrt{1 - x^2}}} \tag{4}$$

For problems see group 5 in the Appendix.

45. Differentiation of u^v.—Let u and v be any two functions of x, and let

$$\mathbf{y = u^v} \tag{1}$$

To differentiate this expression take logarithms of both members and we have

$$\log y = v \log u \tag{2}$$

This is a product of two functions v and $\log u$ and their respective differentials are dv and $\frac{du}{u}$. Therefore, using the method of Art. 32, we may differentiate equation (2), and we have:

$$\frac{dy}{y} = \frac{v \cdot du}{u} + \log u \cdot dv \qquad (3)$$

whence, substituting u^v for y and solving for dy we have

$$\mathbf{dy = vu^{v-1} \cdot du + \log u \cdot u^v \cdot dv} \qquad (4)$$

When u and v are known as functions of x, then du and dv are known in terms of x and dx, and the known values of u, v, du and dv may be substituted in equation (4) thus giving dy in terms of x and dx.

For problems see group 6 in the Appendix.

SUCCESSIVE DIFFERENTIATION AND INTEGRATION.

46. Successive derivatives of a function.—Consider a function of x. For example:

$$y = ax^4 \qquad (a)$$

then

$$\frac{dy}{dx} = 4ax^3 \qquad (1)$$

But this derivative is itself a function of x and its derivative is $12ax^2$; this is also a function of x and its derivative is $24ax$; and so on. These successive derivatives of the original function y are usually represented by the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and so on; so that we may write

$$\frac{d^2y}{dx^2} = 12ax^2 \qquad (2)$$

$$\frac{d^3y}{dx^3} = 24ax \qquad (3)$$

etc. etc.

Equations (1), (2), (3), etc., express what are called the *first derivative*, the *second derivative*, the *third derivative*, etc., of y.

The exact meaning of the successive derivatives of y may be most easily expressed in words when x represents elapsed time. Then $\frac{dy}{dx}$ is the rate of change of y, $\frac{d^2y}{dx^2}$ is the rate of change of $\frac{dy}{dx}$, $\frac{d^3y}{dx^3}$ is the rate of change of $\frac{d^2y}{dx^2}$, and so on. In the particular example, namely $y = ax^4$, the fourth derivative is a constant $(= 24a)$, and the fifth and all higher derivatives are equal to zero.

47. Velocity and acceleration. Use of first and second derivatives.—In the formulation of problems in mechanics one must frequently express the velocity and acceleration of a moving body in terms of its coördinates. Thus x and y are the varying coördinates of a moving particle B in Fig. 19; x and y have definite values at each instant as the body moves, and they are therefore functions of elapsed time; hence we have the following important relations:

$\frac{dx}{dt}$ is the x-component of the velocity of B.

$\frac{dy}{dt}$ is the y-component of the velocity of B.

$\frac{d^2x}{dt^2}$ is the x-component of the acceleration of B.

$\frac{d^2y}{dt^2}$ is the y-component of the acceleration of B.

These relations are evident from the following considerations: Let Δx and Δy be the increments of x and y, respectively, during a short interval of time. Then $\frac{\Delta x}{\Delta t}$ is the average velocity of the body in the direction of the x-axis during the interval Δt, and $\frac{dx}{dt}$ is the limiting value of $\frac{\Delta x}{\Delta t}$ when Δt and Δx approach zero. Similarly $\frac{\Delta y}{\Delta t}$ is the average velocity of the body in the direction

of the y-axis, and $\frac{dy}{dt}$ is the limiting value of $\frac{\Delta y}{\Delta t}$ when Δt and Δy approach zero. Furthermore the x-component of the acceleration

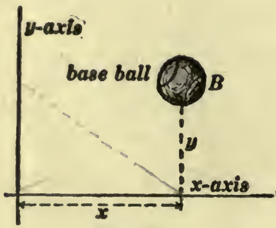


Fig. 19.

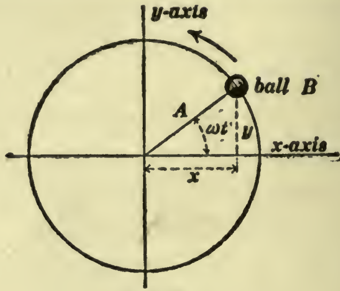


Fig. 20.

of B is equal to the rate of change of the x-component of the velocity of B, that is, the x-component of the acceleration of B is the rate of change of $\frac{dx}{dt}$ which is $\frac{d^2x}{dt^2}$.

Example.—A ball travels at uniform angular velocity (ω radians per second) around a circle of radius A as shown in Fig. 20. In this case the coördinates x and y are simple functions of the elapsed time t; indeed from Fig. 20 we have:

$$x = A \cos \omega t \tag{1}$$
$$y = A \sin \omega t \tag{2}$$

Differentiating these expressions with respect to t we have:

$$\frac{dx}{dt} = -\omega A \sin \omega t \tag{3}$$

and

$$\frac{dy}{dt} = \omega A \cos \omega t \tag{4}$$

These equations express the x and y components of the velocity of the ball B at each instant. Differentiating (3) and (4) with

respect to t we have:

$$\frac{d^2x}{dt^2} = -\omega^2 A \cos \omega t \tag{5}$$

$$\frac{d^2y}{dt^2} = -\omega^2 A \sin \omega t \tag{6}$$

These equations express the x and y components of the acceleration of the ball B at each instant.

PROBLEMS

1. If $x = \tan\theta + \sec\theta$, prove that $\dfrac{d^2x}{d\theta^2} = \dfrac{\cos\theta}{(1-\sin\theta)^2}$.

2. If $y = x^2 \log x$, prove that $\dfrac{d^3y}{dx^3} = \dfrac{2}{x}$.

3. If $y = e^x \log x$, prove that $\dfrac{d^4y}{dx^4} = e^x\left(\log x + \dfrac{4}{x} - \dfrac{6}{x^2} + \dfrac{8}{x^3} - \dfrac{6}{x^4}\right)$.

4. If $y = e^{-x}\cos x$, prove that $\dfrac{d^4y}{dx^4} + 4y = 0$.

5. If $x = A\sin nt + B\cos nt$, prove that $\dfrac{d^2x}{dt^2} + n^2x = 0$.

6. For what values of α will $y = e^{\alpha x}$ satisfy the equation

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0?$$

Ans. $0, -1, +2$.

7. A ball moves so that its x and y coördinates are $At\cos\theta$ and $At\sin\theta$, respectively, as shown in Fig. $p7$, where A and θ are constants and t is elapsed time. Find (a) the x-component

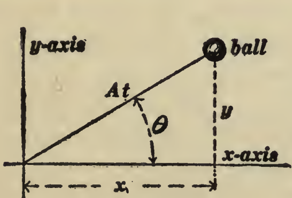


Fig. $p7$.

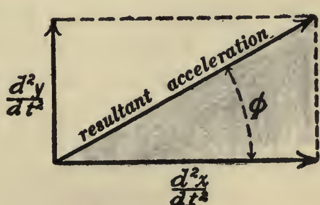


Fig. $p9$.

and (*b*) the y-component of the velocity of the ball, and (*c*) the x-component and (*d*) the y-component of its acceleration.

Ans. (*a*) $A \cos \theta$, (*b*) $A \sin \theta$, (*c*) 0, (*d*) 0

8. A ball moves so that its x and y coördinates are $At \cos \theta$ and $Bt^2 \sin \theta$, respectively, where A, B and θ are constants and t is elapsed time. Find (*a*) the x-component and (*b*) the y-component of the velocity of the ball, and (*c*) the x-component and (*d*) the y-component of its acceleration.

Ans. (*a*) $A \cos \theta$, (*b*) $2Bt \sin \theta$, (*c*) 0, (*d*) $2B \sin \theta$.

9. The resultant acceleration of a body is $\sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$ as shown in Fig. $p9$, where ϕ is the angle whose tangent is $\dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$.

Show from equations (1) to (6) of Art. 47 that the resultant acceleration (or total acceleration) of the ball B in Fig. 20 is equal to $\frac{v^2}{A}$, and show that it is parallel to the radius A at each instant, v being the resultant velocity $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

10. If the distance traveled by a body is $s = ae^{\alpha t} + be^{-\alpha t}$, show that the acceleration is $\alpha^2 s$. If t is expressed in seconds and s in feet, (*a*) in what units is α expressed, and (*b*) in what units is $\alpha^2 s$ expressed?

Ans. (*a*) reciprocal seconds, (*b*) feet per second per second.

48. Harmonic motion. Example showing the use of a second derivative.—A body m is suspended by a spring and the body stands in equilibrium in the position shown in Fig. 21. If the body is displaced x feet downwards (or upwards) an unbalanced force F will act upon the body, and this unbalanced force will be proportional to x. Therefore we may write

$$F = -kx \tag{1}$$

The negative sign is chosen because F is upwards when x is downwards, and F is downwards when x is upwards.

If the body is pulled upwards or downwards and released, it will vibrate up and down, and x will vary with the time t. Then $\frac{dx}{dt}$ is the velocity of the body at each instant, and $\frac{d^2x}{dt^2}$ (the rate of change of $\frac{dx}{dt}$) is the acceleration of the body at each instant. But the unbalanced force which acts upon a body is equal* to *mass* $\times$ *acceleration*. Therefore $F = m \cdot \frac{d^2x}{dt^2}$, or, substituting the value of F from equation (1), we have:

$$\frac{d^2x}{dt^2} = -\frac{k}{m} \cdot x \qquad (2)$$

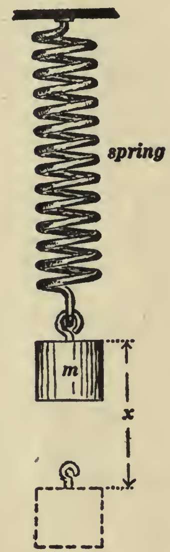


Fig. 21.

To determine the motion of the body in Fig. 21 it is necessary to find x as a function of t which will satisfy the law of growth as expressed by equation (2), and the problem is solved if we find a function of t whose second derivative is equal to $-\frac{k}{m}$ multiplied by the function itself. This function would be at once recognized if one were familiar with the derivatives of all kinds of functions. Thus we have another example showing how important it is to be familiar with the derivatives of functions. Compare Arts. 22 and 23.

Consider the function

$$x = a \sin (\omega t + \theta) \qquad (3)$$

where a, ω and θ are undetermined constants. Differentiating†

* When mass is expressed in pounds, acceleration in feet per second per second, and force in poundals; or when C.G.S. units are used throughout.

† Let $z = \omega t + \theta$ and use the formula for the differentiation of a function of a function as explained in Art. 34.

equation (3) with respect to t we have:

$$\frac{dx}{dt} = \omega a \cos(\omega t + \theta) \tag{4}$$

and differentiating again, we have:

$$\frac{d^2x}{dt^2} = -\omega^2 a \sin(\omega t + \theta) \tag{5}$$

or, substituting x for $a \sin(\omega t + \theta)$, we have:

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{6}$$

Therefore, if

$$\omega^2 = \frac{k}{m} \tag{7}$$

equation (3) gives the required function, and a and θ can have any values whatever.

If the vibrations are started by drawing the body down to $x = A$, and if time is reckoned from the instant of release, then $x = A$ when $t = 0$, and also $\frac{dx}{dt} = 0$ when $t = 0$ because the body has no velocity at the instant of release. Using the values $\frac{dx}{dt} = 0$ and $t = 0$ in equation (4), we find $\theta = 0$. Therefore equation (3) becomes:

$$x = a \sin \omega t \tag{8}$$

Using the values $x = A$ and $t = 0$ in equation (8), we find $a = A$. Therefore, *for the case in which the body is pulled down to $x = A$ and released when $t = 0$,* equation (8) becomes:

$$x = A \sin \omega t \tag{9}$$

The motion which is defined by equation (9), or in general by equation (3), is called *simple harmonic motion.* The distance x

of the vibrating body from its equilibrium position is at each instant proportional to the sine of a uniformly growing angle ωt; or, the position P of the vibrating body is at each instant the projection on the straight line ab of a point P' which moves round the circle CC at constant angular velocity ω, the radius of the circle being A, as shown in Fig. 22.

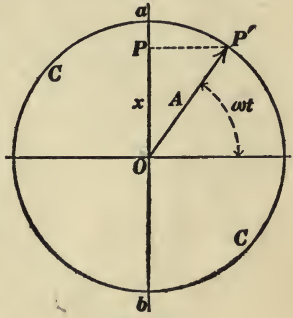


Fig. 22.

One complete vibration of the body in Fig. 21 takes place while the point P' in Fig. 22 makes one complete revolution, that is, while the angle ωt increases from zero to 2π radians, or while the time t increases from zero to $\frac{2\pi}{\omega}$ seconds, ω being expressed in radians per second. Therefore $\frac{2\pi}{\omega}$, or $2\pi \div \sqrt{\frac{k}{m}}$ [see equation (7)], is the *period* of one complete vibration. That is $2\pi \div \sqrt{\frac{k}{m}}$ is the number of seconds per vibration, or $\sqrt{\frac{k}{m}} \div 2\pi$ is the number of vibrations per second, or *frequency*, of the vibrating body.

In this discussion the mass of the spring and the effects of friction are ignored.

49. Curvature.—Consider the curve cc in Fig. 23. The tangent line a touches the curve at p and the tangent line b touches the curve at q. Therefore the tangent line turns through the angle β when the point of tangency moves from p to q. The sharper the bend or curvature of the line cc the greater the value of β for a given length of arc pq. Dividing the angle β (expressed in radians) by the length of the arc pq we have a quotient which is used as a measure of the *average curvature* of cc between p and q, and the limiting value of this quotient when

the arc pq approaches zero is the measure of the *curvature of* cc *at the point* p.*

It is required to find an expression for the limiting value of the quotient $\frac{\text{angle } \beta}{\text{arc } pq}$ in Figs. 23 and 24 when the arc pq approaches zero. Let the coördinates of p be x and y, and let the coördi-

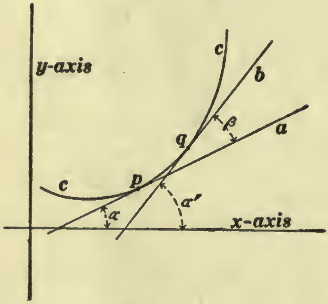


Fig. 23.

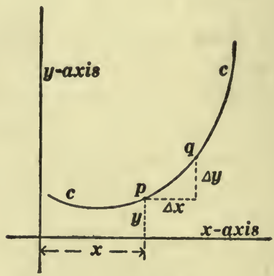


Fig. 24.

nates of q be $x + \Delta x$ and $y + \Delta y$ as shown in Fig. 24. Then:

$$\text{chord } pq = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{1}$$

and when the arc pq approaches zero the arc becomes more and more nearly equal to the chord so that the limiting form of equation (1) is:

$$\text{arc } pq = \sqrt{(dx)^2 + (dy)^2} \tag{2}$$

Furthermore we have:

$$\tan \alpha = \frac{dy}{dx} \tag{3}$$

according to Art. 18, where α is the angle shown in Fig. 23; and of course $\tan \alpha'$ (see Fig. 23) is the value of $\frac{dy}{dx}$ at the point q. But $\frac{dy}{dx}$ changes $\frac{d^2y}{dx^2}$ times as fast as x, and therefore the change

* Or at q, because p and q approach coincidence as the arc approaches zero.

in the value of $\frac{dy}{dx}$ from p to q is $\frac{d^2y}{dx^2} \cdot dx$, when Δx is infinitely small. Therefore the value of $\frac{dy}{dx}$ at q is $\frac{dy}{dx} + \frac{d^2y}{dx^2} \cdot dx$, and consequently we have:

$$\tan \alpha' = \frac{dy}{dx} + \frac{d^2y}{dx^2} \cdot dx \tag{4}$$

Now the angle β in Fig. 23 is equal to $\alpha' - \alpha$, and, since $\alpha' - \alpha$ is an infinitesimal angle, we may write:

$$\tan (\alpha' - \alpha) = \alpha' - \alpha$$

Therefore we have:

$$\beta = \alpha' - \alpha = \tan (\alpha' - \alpha) \tag{5}$$

But

$$\tan (\alpha' - \alpha) = \frac{\tan \alpha' - \tan \alpha}{1 + \tan \alpha' \tan \alpha} \tag{6}$$

Therefore, substituting the values of $\tan \alpha$ and $\tan \alpha'$ from equations (3) and (4), and writing β for $\tan (\alpha' - \alpha)$ according to equation (5), we have:

$$\beta = \frac{\frac{d^2y}{dx^2} \cdot dx}{1 + \frac{dy}{dx}\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} \cdot dx\right)} \tag{7}$$

But the infinitesimal term in the denominator $\left(\frac{d^2y}{dx^2} \cdot dx\right)$ may be discarded because it is added to a finite quantity $\left(\frac{dy}{dx}\right)$. Therefore equation (7) becomes:

$$\beta = \frac{\frac{d^2y}{dx^2} \cdot dx}{1 + \left(\frac{dy}{dx}\right)^2} \tag{8}$$

Therefore, using the value of arc pq from equation (2) we have

$$\frac{\beta}{\operatorname{arc} pq} = \frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2} \cdot \frac{dx}{\sqrt{(dx)^2+(dy)^2}} \tag{9}$$

This is the required limiting value of $\dfrac{\beta}{\operatorname{arc} pq}$ because equations (2) and (5) are limiting forms and because an infinitesimal has been discarded in going from equation (7) to equation (8). Dividing numerator and denominator of $\dfrac{dx}{\sqrt{(dx)^2+(dy)^2}}$ by dx this factor becomes $\dfrac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}$ and equation (9) becomes:

$$\text{Curvature at } p = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{10}$$

Curvature of a circle.—The idea of curvature is necessarily vague until we understand clearly the unit in terms of which curvature is expressed. Therefore let us apply the above definition of curvature to a circle. The line a, Fig. 25, is tangent to the circle at p and b is tangent to the circle at q, and the measure of the curvature of the circle is $\dfrac{\text{angle } \beta}{\operatorname{arc} pq}$. But the angle β, between the tangent lines is equal to the angle between the radii Cp and Cq. Therefore the angle β, in radians, is equal to arc pq divided by the radius r of the circle, and consequently the quotient

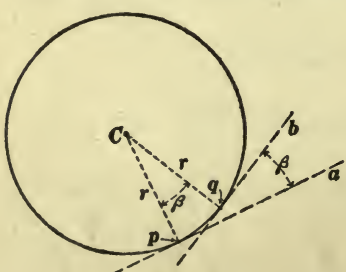


Fig. 25.

$\dfrac{\text{angle } \beta}{\text{arc } pq}$ is equal to $\dfrac{1}{r}$. That is, the curvature of a circle is measured by the reciprocal of its radius, or the reciprocal of the curvature of a circle is equal to the radius of the circle. *Therefore the reciprocal of the curvature of any curve at a point is called the radius of curvature of the curve at the point.* In accordance with this statement equation (10) may be written:

$$\left\{\begin{matrix}\text{radius of curvature of any} \\ \text{curve at a point}\end{matrix}\right\} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \tag{11}$$

the coördinates of the point p being substituted in the general expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Remark.—At a point where a curve is horizontal (parallel to the x-axis) the first derivative $\dfrac{dy}{dx}$ is equal to zero, and at such a point the expression for radius of curvature reduces to $\dfrac{1}{\frac{d^2y}{dx^2}}$. Where a curve is very nearly horizontal the first derivative $\dfrac{dy}{dx}$ is small as compared with unity, the expression $1 + \left(\dfrac{dy}{dx}\right)^2$ is sensibly equal to unity, and the radius of curvature is sensibly equal to $\dfrac{1}{\frac{d^2y}{dx^2}}$.

Example.—Consider the parabola whose equation is $y = x^2$. Then $\dfrac{dy}{dx} = 2x$ and $\dfrac{d^2y}{dx^2} = 2$. To find the radius of curvature at the origin of coördinates, substitute the values $\dfrac{dy}{dx} = 2x = 0$ and

6

$\frac{d^2y}{dx^2} = 2$ in equation (11) which gives $R = 2$. To find the radius of curvature at the point $x = 1$ and $y = 1$, substitute the values $\frac{dy}{dx} = 2x = 2$ and $\frac{d^2y}{dx^2} = 2$ in equation (10) which gives $R = 5.6$.*

Further discussion of curvature. The osculating circle.—A straight line may *coincide* with a curve at a point and *have the same direction* as the curve at the point. Such a straight line is called a *tangent line.*

A circle may *coincide* with any curve at a point, *have the same direction* as the curve at the point, and *have the same curvature* as the curve at the point. To coincide with the curve at a point means that the ordinate y of the circle is equal to the ordinate y of the curve for a certain abscissa x. To have the same direction as the curve at the point means that $\frac{dy}{dx}$ for the circle is equal to $\frac{dy}{dx}$ for the curve for the same value of x. To have the same curvature as the curve at the point means that $\frac{d^2y}{dx^2}$ for the circle is equal to $\frac{d^2y}{dx^2}$ for the curve (if the condition of tangency is satisfied) for the same value of x. The circle which satisfies these three

* Whenever an equation between x and y is to be plotted as a curve, and whenever definite values are to be determined for the derivatives of y with respect to x, **the question arises as to what units are to be used in the expression of y and its various derivatives.** Quantities such as 2 *inches*, 5 *pounds* and 6 *hours*, are called *denominate quantities.* The ratio of two denominate quantities of the same kind is called a *pure number.* Both members of an equation and every separate term in each member must have the same denomination. Consider the equation $y = ax^2$. In order to plot this equation as a curve, x and y must both be expressed in inches, let us say. Therefore the coefficient a is a denominate quantity which gives *inches* when it is multiplied by *inches squared.* That is, the reciprocal of a is expressed in inches. When an equation such as $y = 4x^3$ is to be plotted as a curve, x and y must both be expressed in inches, and the coefficient 4 must be thought of as a denominate number, the denomination, or unit, being such as will give y in inches when x is expressed in inches.

conditions at a point on a curve is called the *osculating circle* at the point, and the radius of the osculating circle is called the *radius of curvature* of the curve at the point.

Determination of the osculating circle.—One can easily find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at a given point p on a given curve cc (see Fig. 26) by differentiating the equation of the curve.

Consider any circle whatever as shown in Fig. 27. The equation

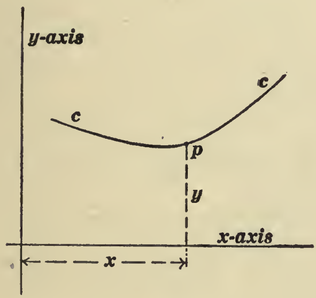


Fig. 26.

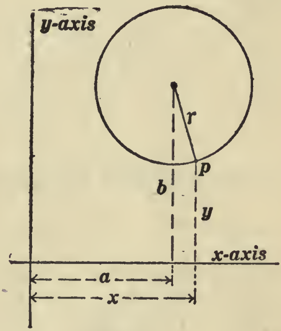


Fig. 27.

of this circle is

$$(x - a)^2 + (y - b)^2 = r^2 \tag{12}$$

where a and b are the coördinates of the center of the circle and r is the radius of the circle. Differentiating equation (12), we have

$$2(x - a) \cdot dx + 2(y - b) \cdot dy = 0$$

or

$$(x - a) + (y - b)u = 0 \tag{13}$$

where u is written for $\frac{dy}{dx}$. Differentiating equation (13), remembering that y and u are both functions of x, we have:

$$dx + (y - b) \cdot du + u \cdot dy = 0$$

Dividing through by dx and remembering that $u \cdot \dfrac{dy}{dx} = u^2$, we have:

$$1 + (y - b) \cdot \frac{du}{dx} + u^2 = 0$$

But $\dfrac{du}{dx}$ is the second derivative $\dfrac{d^2y}{dx^2}$. Therefore, representing $\dfrac{d^2y}{dx^2}$ by v, we have:

$$1 + (y - b)v + u^2 = 0$$

or

$$y - b = -\frac{1 + u^2}{v} \tag{14}$$

and, substituting this value of $(y - b)$ in equation (13), we have:

$$x - a = \frac{(1 + u^2)u}{v} \tag{15}$$

Substituting the values of $(x - a)$ and $(y - b)$ from equations (15) and (14) in equation (12), we get:

$$r = \frac{(1 + u^2)^{\frac{3}{2}}}{v} \tag{16}$$

Now u and v and x and y are all known *at a given point* p *of the curve* cc of Fig. 26, and these four quantities u, v, x and y have the same values *for the osculating circle* where it touches the curve cc at the point p. Therefore a and b can be found from equations (14) and (15), and r can be determined from equation (16) so that the osculating circle at the point p is completely determined.

PROBLEMS.

1. Find the radius of curvature at the point p of the parabola shown in Fig. $p1$.

Ans. 40.5 inches.

2. Find the radius of curvature of the ellipse shown in Fig. $p2$, (a) at the point p, and (b) at the point p'.

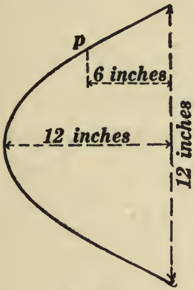


Fig. $p1$.

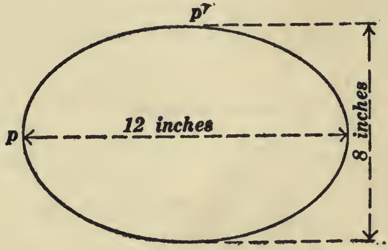


Fig. $p2$.

Ans. (a) 2.67 inches, (b) 18 inches.

Note.—The value of $\frac{dy}{dx}$ is infinite where a curve is parallel to the y-axis and generally the value of $\frac{d^2y}{dx^2}$ is also infinite at such a point. Therefore the expression for radius of curvature at such a point of a curve usually becomes indeterminate. It is evident, however, that x and y may be interchanged throughout the entire discussion of Art 49. This interchange gives an expression for radius of curvature in terms of $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$ where x is thought of as a function of y; and this expression can be used to determine the radius of curvature at a point on the curve where the curve is parallel to the y-axis.

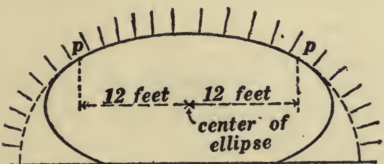


Fig. $p3$.

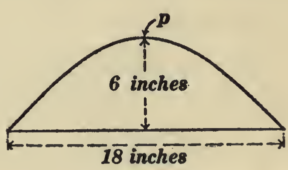


Fig. $p5$.

3. A stone arch is to be made as shown in Fig. $p3$, the two dotted circles having the same curvature as the ellipse at the junction points pp. Find the radius of the circles.

Ans. 14.07 feet.

4. Find the radius of curvature of the curve $y = \sin x$ at the points (*a*) $x = 0$ and (*b*) $x = \frac{\pi}{4}$.

Ans. (*a*) infinity, (*b*) 2.6.

5. Find the radius of curvature at the point p of the sine curve shown in Fig. *p*5.

Ans. 5.47 inches.

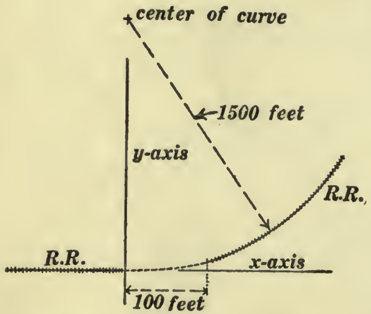


Fig. *p*6.

6. A straight portion of railway track merges gradually into a circular curve as indicated by the dotted line in Fig. *p*6. The dotted portion is called a *transition curve.* The curvature of the track (which is sensibly equal to $\frac{d^2y}{dx^2}$ because $\frac{dy}{dx}$ is very nearly zero) increases uniformly from zero at $x = 0$ to $\frac{1}{1500}$ at $x = 100$. Find the equation of the dotted transition curve.

Ans. $900{,}000\ y = x^3$.

Note.—If the curvature $\frac{d^2y}{dx^2}$ increases uniformly its rate of change, $\frac{d^3y}{dx^3}$, is a constant.

50. Geometric differentiation. The acceleration of a particle which travels in a circular orbit.—There are cases in which the instantaneous rate of change of a given quantity may be found when the quantity is specified in geometric terms. An important example is the following: *A particle travels round and round a circular path or orbit at velocity* v, *and it is required to find the acceleration* $\frac{dv}{dt}$ *of the particle.*

At a certain instant the particle is at P, Fig. 28, and its velocity

at this instant is represented in direction and magnitude by the line v_1. During a short interval of time the particle will travel

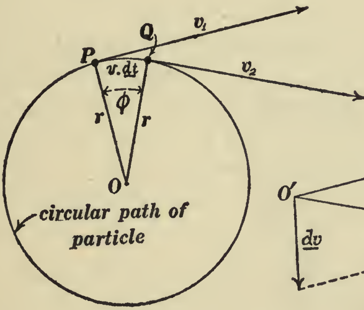


Fig. 28.

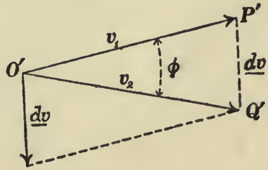


Fig. 29.

the distance $v.dt$, the particle will then be at Q, and its velocity will be as represented by the line v_2 *which has the same length as* v_1.

From any fixed point O' draw two lines parallel and equal to v_1 and v_2 respectively, as shown in Fig. 29. Then the line dv represents the velocity which must be added* to v_1 to give v_2, that is the line dv represents the increment of the velocity of the particle during the interval dt.

Now when dt is chosen smaller and smaller, the angle ϕ approaches zero, and the angles at P' and Q' approach 90° so that dv becomes more and more nearly parallel to the radius OP. Also when dt is very small the arc PQ $(= v \cdot dt)$ is sensibly a straight line and the two triangles OPQ and $O'P'Q'$ are similar. Therefore from these similar triangles we have:

$$r : v.dt : : v : dv \tag{1}$$

In this proportion v is written instead of v_1 or v_2, both of which

* We are here concerned with what is called *vector addition* or *geometric addition*. The most familiar example of this kind of addition is the addition of two forces by the principle of the "parallelogram of forces."

have the same numerical value. Solving this proportion we have:

$$\frac{dv}{dt} = \frac{v^2}{r} \tag{2}$$

We have already proceeded to the limit in considering the arc PQ a straight line. That is, the acceleration of a particle moving in a circular orbit is equal to the square of the velocity of the particle divided by the radius of the orbit, and the acceleration is at each instant towards the center of the circle because the infinitesimal increment of velocity dv in Fig. 29 is in the direction of PO in Fig. 28.

51. Geometric differentiation. The inward force per unit length of a barrel hoop.—Another important case of geometric differentiation is the following: Figure 30 represents a hoop around a tank. *It is required to find the value of* $\frac{dF}{ds}$ *where* dF *is the force with which a short length* ds *of the hoop pushes inwards on the tank.*

The radius r of the tank and the tension T of the hoop are given. The tension of the hoop is the force with which a portion of the hoop pulls on an adjoining portion. Thus the two forces

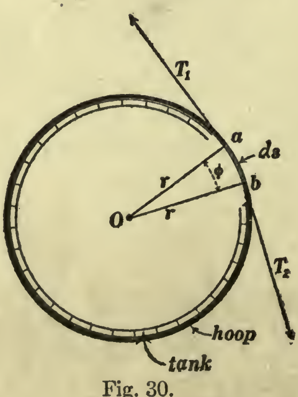


Fig. 30.

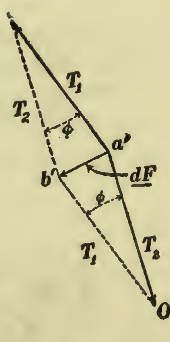


Fig. 31.

T_1 and T_2, Fig. 30, which pull on the portion ds of the hoop in Fig. 30, are each equal to the tension T, and the resultant of the two forces T_1 and T_2 is the total inward force dF exerted on the portion; and this total inward force acting on ds is the force with which ds pushes against the tank.

The resultant of T_1 and T_2 is shown in Fig. 31, and the triangle $a'O'b'$ in Fig. 31 is similar to the triangle aOb in Fig. 30. Therefore we have

$$r : ds : : T : dF$$

so that

$$\frac{dF}{ds} = \frac{T}{r} \tag{1}$$

We have already proceeded to the limit in considering aOb to be a triangle, that is, in considering ds to be a straight line, which is only true when ds is infinitely small.

The practical meaning of equation (1) can be best understood by thinking of $\frac{dF}{ds}$ as the really important quantity under consideration, *not* by thinking of the meaning of F.*

* Indeed one gets into unfamiliar vector theory when one tries to consider the meaning of F.

CHAPTER III.

INTEGRATION.

52. Integration as an arithmetical argument.—Any argument which leads to a proposition concerning *total change* when a *rate of change* is given may properly be called *integration*. Thus the following arguments are the simplest existing examples of integration:

(*a*) A man earns money at the constant rate of two dollars per day. Therefore the man earns two dollars each day, and in 100 days he would earn 200 dollars. Or, to get the amount of money earned in 100 days multiply two dollars per day by 100 days.*

(*b*) One man A earns money twice as fast as another man B. Therefore in any interval of time whatever A earns twice as much as B.

53. Integration as a mechanical process. Integrating machines.—The simplest and most familiar integrating machine is the ordinary *cyclometer*. The wheel of a bicycle turns at a speed which is proportional to the velocity of travel of the bicycle. Therefore the number of revolutions of the wheel is proportional to the distance traveled, and the cyclometer is a revolution counter arranged to read one additional unit for each mile traveled.

The ordinary "electricity meter"† which is installed in connection with electric lamps is an integrating machine. The

* If one can get 200 dollars by multiplying two dollars per day by 100 days, one might well ask how to perform such a profitable operation? The answer is: Work hard for 100 days! That is what this particular multiplication means. Nearly every mathematical process rightly understood relates to physical reality in a manner as definite and as exacting as this particular case of multiplication.

† Properly called a *watt-hour* meter.

spindle of the meter turns at a speed which is proportional to the *rate of delivery of energy* (in watts) to a customer. Therefore the number of revolutions of the spindle is proportional to the total amount of energy delivered. The clock-work of the watt-hour meter is a revolution counter which is arranged to read one additional unit for each watt-hour of energy delivered.

54. The planimeter.*—The planimeter is an integrating machine for measuring area, and it depends upon the following kinematical principle:

Consider a line AB, Fig. 32, moving in any manner whatever in the plane of the paper. *The motion of the line at each instant may be thought of as* **a motion of translation** *combined with* **a motion of rotation about an arbitrary point** p.†

Let v be the component at right angles to AB of the velocity of translation, as shown in Fig. 32, and let ω be the angular velocity of AB about the point p. It is desired to find the rate at which the line AB sweeps over area, *area swept over by the line being considered as positive when the line sweeps over it from right to left to an observer looking from A towards B.*

During a short interval of time, Δt, the motion of translation carries the line from AB to $A'B'$ as shown in Fig. 34, and the

* The planimeter which is here described is the Amsler planimeter.

The student wishing to become familiar with various kinds of integrating machines should consult *Les Integraphes* Abdank-Abakanowicz, Paris, Gauthier-Villars.

A discussion of integrating machines is also given in *Encyclopädie der Mathematischen Wissenschaften*, Vol. II. This great work is now appearing in a revised form in a French translation.

See also *Report on Planimeters, British Association Annual Report*, 1894, pages 496–523.

See also *Mechanical integrators*, Shaw, *Proceedings of the Institution of Civil Engineers*, London, 1885, pages 75–143.

The earliest type of integrating machine for use in harmonic analysis is that of Lord Kelvin. This machine is described in Franklin's *Electric Waves*, pages 240–242, The Macmillan Co., New York, 1909.

† See Franklin and MacNutt's *Mechanics and Heat*, Arts. 74–77, pages 174 and 175, The Macmillan Co., N. Y., 1910.

area swept over by the line is $l \times v \cdot \Delta t$. Dividing this area by Δt gives the rate at which the line sweeps over area because of its motion of translation.

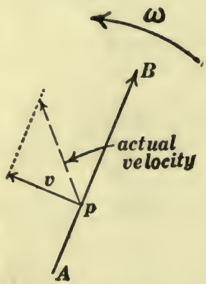


Fig. 32.

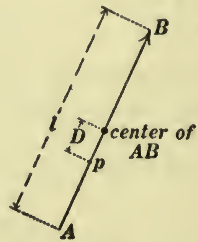


Fig. 33.

During a short interval of time, Δt, the motion of rotation carries the line from AB to $A''B''$, as shown in Fig. 35, and the

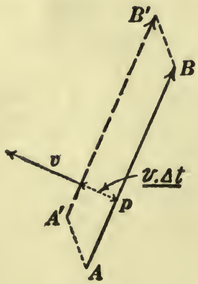


Fig. 34.

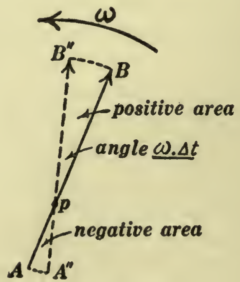


Fig. 35.

area swept over consists of two parts as shown. The positive area in Fig. 35 is equal to $\frac{1}{2}\left(\frac{l}{2} + D\right)^2 \times \omega \cdot \Delta t$, the negative area in Fig. 35 is equal to $\frac{1}{2}\left(\frac{l}{2} - D\right)^2 \times \omega \cdot \Delta t$, and the total area swept over is equal to $lD\omega \cdot \Delta t$. Dividing this area by Δt gives the rate at which the line sweeps over area because of its motion of rotation.

Therefore the total rate at which the line sweeps over area $\left(\frac{dA}{dt}\right)$ is equal to $lv + lD\omega$, or, using $\frac{d\theta}{dt}$ for ω, we have

$$\frac{dA}{dt} = lv + lD\frac{d\theta}{dt} \tag{1}$$

Let AB be a metal bar carrying a wheel mounted as shown in Fig. 36 (which is a side view), and let the rim of this wheel roll on the paper as the bar AB moves, as shown in Figs. 36 and 37.

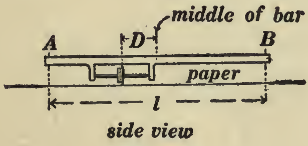


Fig. 36.

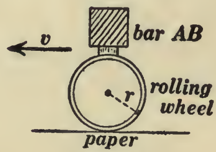


Fig. 37.

Let ψ be the total angle turned by the wheel in a given time; then $\frac{d\psi}{dt}$ is the angular velocity of the rolling wheel, and it is equal to $\frac{v}{r}$, where r is the radius of the wheel, as shown in Fig. 37. Therefore we may substitute $r\frac{d\psi}{dt}$ for v in equation (1) and we have:

$$\frac{dA}{dt} = lr\frac{d\psi}{dt} + lD\frac{d\theta}{dt} \tag{2}$$

in which $lr\frac{d\psi}{dt}$ is the rate of sweeping over area because of the translatory motion of AB, and $lD\frac{d\theta}{dt}$ is the rate of sweeping over area because of the rotatory motion of AB.

Now if the rate of sweeping area due to the translatory motion of AB is lr times the rate of turning of the wheel, then the total area swept over because of the translatory motion of AB is lr times the total angle ψ turned by the wheel.

Similarly, if the rate of sweeping area due to the rotatory motion

of AB is lD times the rate of turning of AB, then the total area swept over because of the turning of AB is lD times the total angle θ turned by AB.

Therefore the total area swept over by AB is $lr\psi + lD\theta$.* That is,

$$A = lr\psi + lD\theta \tag{3}$$

If the bar AB comes back to its initial position without turning completely round, then $\theta = 0$ and equation (3) becomes

$$A = lr\psi \tag{4}$$

That is, the total area swept over by AB is proportional to the angle ψ turned by the rolling wheel, and the value of lr may be so chosen that one square inch corresponds to each revolution of the wheel. In this case square inches of area are indicated by whole revolutions of the wheel, and the circumference of the wheel may be divided so as to indicate fractions of a square inch.

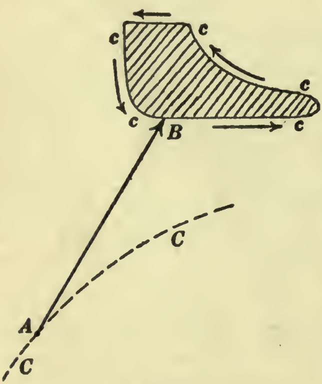


Fig. 38.

In the actual planimeter one end of the bar AB is constrained to move back and forth along a circle† CC, Fig. 38, while the other end A travels once round the closed curve cc which bounds the area to be measured. Then any area outside of cc which is swept over by AB at all is swept over as many times to the left as to the right, but every portion of the shaded area is swept over once more to the left than to the right. Therefore,

* How much more easily understood is this argument than to say *integrating equation (2) we have equation (3)*! and yet this brief statement means exactly what is given in the above argument.

† Along a straight line in some forms of planimeter.

in view of the agreement as to algebraic signs, the total area $A\ (= lr\psi)$ swept over by the line AB is equal to the shaded area.

55. Integration by steps.—In the laying out of a new electric railway the engineer usually makes a study of the probable schedule speed of the car in order to be able to estimate the traffic income and to be able to choose suitable driving motors. In the

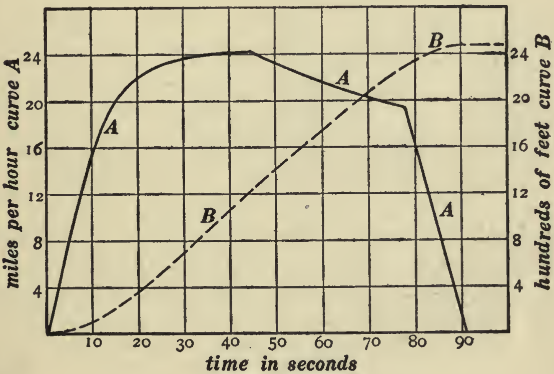


Fig. 39.

study of a particular run of a car, the curve A, Fig. 39, is determined from the known characteristics of a motor* (tentatively chosen), the weight of the car, and the curvature and grade of the track. The curve A gives the speed of the car as a function of elapsed time and it is called a *speed-time curve.* In the particular case represented in Fig. 39 the car is being accelerated from 0 to 45 seconds; at 45 seconds the power is turned off and the car is allowed to coast; and at 78 seconds the brakes are applied and the car is rapidly brought to rest. The ordinate of curve A is the velocity v of the car, the product $v \cdot dt$ is the distance traveled during the infinitesimal interval dt, and $\int_{t=0}^{t=t} v \cdot dt$ is the total distance traveled by the car from the start up to any given instant

* These characteristics are furnished by the manufacturing company.

t. This distance is shown as a function of t in the accompanying

SPEED AND DISTANCE TABLE OF AN ELECTRIC STREET CAR

(See Fig. 39).

Time in Seconds.	Speed: Miles Per Hour.	Distance in Feet.
0	0	0
10	15.2	111
20	22.0	384
30	23.6	719
40	24.2	1,071
45	24.3	1,246
60	21.6	1,751
78	19.4	2,292
91	0	2,476

table and also it is shown by the ordinate of the curve BB in Fig. 39. The tabulated values of the distance traveled by the car are calculated approximately as follows: From 0 to 10 seconds the average speed of the car is approximately $\frac{0 + 15.2}{2}$ miles per hour, and the distance traveled during this 10-second interval is found by multiplying the average speed (reduced to feet per second) by 10 seconds which gives 111 feet. From 10 seconds to 20 seconds the average speed of the car is approximately $\frac{15.2 + 22.0}{2}$ miles per hour, and the distance traveled during this 10-second interval is found by multiplying the average speed (reduced to feet per second) by 10 seconds, which gives 273 feet, so that the total distance traveled during 20 seconds is 111 feet + 273 feet, which is 384 feet; and so on.

PROBLEMS.

1. The accompanying figure, $p1$, shows the stretching force per square inch of section and the fractional elongation of a sample of steel under test. The sample is one inch in diameter and the portion which is stretched is 4 inches long initially. Find

the work in foot-pounds expended on the specimen up to the breaking point b.

Ans. 53.1 foot-pounds.

Note.—Multiplying abscissas in Fig. $p1$ by 4 inches gives actual elongations in inches, and multiplying ordinates by square inches of section of sample gives actual stretching forces in pounds. Let elongation in inches be e and let stretching force be F. Then work done is the integral of $F \cdot de$.

2. Find the work (in ergs and in foot-pounds) required to magnetize a wrought iron bar 3 inches by 3 inches by 20 inches

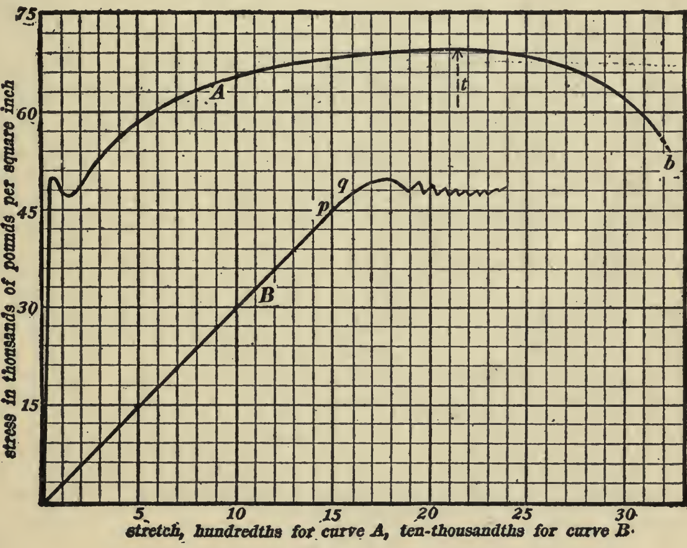


Fig. $p1$.

from a neutral condition to $B = 16{,}000$ lines per square centimeter.

Ans. 326×10^5 ergs or 2.406 foot-pounds.

Note.—The work done in magnetizing iron is:

$$W = \frac{V}{4\pi}\int H \cdot dB$$

where W is expressed in ergs and V is the volume of the iron in cubic centimeters. There are $30.5 \times 453.6 \times 980$ ergs in one foot-pound.

TABLE OF B AND H FOR WROUGHT IRON.

H	B
10	12,400
20	14,330
30	15,100
40	15,550
50	15,950
60	16,280
70	16,500

3. The force F which acts on a body is given as a function of the time in the accompanying table. Force being expressed in poundals and time in seconds. The mass of the body is 1,000 pounds. Find the velocity produced from $t = 0$ to $t = 40$ seconds.

Ans. 27.1 feet per second.

Note.—The acceleration $\frac{dv}{dt} = \frac{F}{m}$ or $dv = \frac{F}{m} \cdot dt$ or the velocity v is equal to $\frac{1}{m}$ times the integral of $F \cdot dt$; velocity being expressed in feet per second.

TABLE OF F AND t.

t in Seconds.	F in Poundals.
0	500
10	625
20	700
30	745
40	785

56. Algebraic integration.—Integration, in the sense in which this word is used throughout this text, is the recognition of a given differential expression as the differential of a known function, and the rules for differentiation which are given in Chapter II are rules for integration also. Thus the differential of $\log x + C$ is $\frac{dx}{x}$, and therefore, by definition, the integral of $\frac{dx}{x}$ is $\log x + C$,

where C is any constant. The rules given in Chapter II cover differentiation completely, or nearly so, but they do not cover integration completely. Indeed complete rules cannot be given for integration.*

In the recognition of a given differential expression as the differential of a known function, two things are helpful, namely, (*a*) A table of standard forms or functions with their differentials (such a table is usually called a *table of integrals*), and (*b*) some degree of familiarity with the transformations which may be used to reduce a given differential expression to a standard form or to a combination of standard forms.

A table of integrals is given in Appendix B.

57. Transformation of differential expressions.—When a differential equation has been found by differentiating a given function, the integral of the differential equation is of course known. Many of the forms in the table of integrals have been established in this way as problems in differentiation in Chapter II. When a given differential equation has not been found by differentiating a known function it is in many cases† possible to *invent* a scheme for transforming the given differential expression so as to reduce it to one or more of the standard forms in the table of integrals. The most important‡ of these transformations are embodied in the following rules, and a few of the lesser important transformations are illustrated by problems.

As a very simple example consider the differential equation

* Integration by series is a universal and systematic method of integration. Any given algebraic expression for $\frac{dy}{dx}$ can be expanded by Maclaurin's theorem, each term of the series so obtained can be integrated according to form 1 in the table of integrals, and the resulting series, *when it is convergent* is the integral of the given expression for $\frac{dy}{dx}$.

† Not in every case, by any means; many integrals *cannot* be expressed in terms of the elementary functions which are used in the table of integrals.

‡ Transformations depending upon the use of imaginaries are also very important. See Art. 93 in Chap. VI.

which was found in the discussion of the arc of a parabola in Art. 20, namely:

$$ds = \sqrt{1 + 4k^2x^2} \cdot dx \tag{1}$$

This expression is easily reduced to:

$$ds = 2k\sqrt{\frac{1}{4k^2} + x^2} \cdot dx \tag{2}$$

and according to rule I, below, we have:

$$\int 2k\sqrt{\frac{1}{4k^2} + x^2} \cdot dx = 2k\int \sqrt{\frac{1}{4k^2} + x^2} \cdot dx \tag{3}$$

Therefore, writing a^2 for $\frac{1}{4k^2}$, we get an expression which is identical to form 57 in the table of integrals.

Rule I.—Any constant factor may be removed from under the integral sign. Thus:

$$\int a \cdot du = a\int du$$

where a is a constant and u is any differential expression whatever.

Rule II.—The integral of the sum of two differential expressions is equal to the sum of the integrals of the respective expressions. Thus:

$$\int (du + dv) = \int du + \int dv$$

where du and dv are any differential expressions whatever.

Rule III.—When a differential expression is of the form $u^n \cdot du$ its integral is, of course, given by form 1 or by form 2 of the table of integrals, whatever the function u may be.

For example consider the differential equation:

$$dy = \sin^2 x \cos x \cdot dx \tag{4}$$

Let $\sin x = u$ then $\sin^2 x = u^2$ and $\cos x \cdot dx = du$, and equation (4) becomes:

$$dy = u^2 \cdot du$$

whence

$$y = \tfrac{1}{3}u^3 + C$$

or, substituting $\sin x$ for u we have.

$$y = \tfrac{1}{3}\sin^3 x + C \tag{5}$$

For problems see group 7 in the appendix.

Rule IV. Integration by parts.—This rule is expressed thus:

$$\int u \cdot dv = uv - \int v \cdot du \tag{6}$$

This formula is sometimes useful when a given differential expression can be resolved into two factors, namely, (*a*) any function u, and (*b*) a familiar differential expression dv.

This rule is based on Art. 32 where it is shown that:

$$d(uv) = u \cdot dv + v \cdot du$$

which is to say:

$$uv = \int u \cdot dv + \int v \cdot du$$

so that, by transposing we get equation (6).

For example consider the differential equation:

$$dy = x \log x \cdot dx \tag{7}$$

In this case $x \cdot dx (= dv)$ is a familiar differential expression, that is, it is the differential of $\tfrac{1}{2}x^2 (= v)$. Therefore, using equation (6), we have:

$$\int u \cdot dv = u \cdot v - \int v \cdot du \tag{6}$$

$$\int (\log x)(x \cdot dx) = (\log x)(\tfrac{1}{2}x^2) - \int (\tfrac{1}{2}x^2)\left(\frac{dx}{x}\right) \tag{8}$$

But $(\tfrac{1}{2}x^2)\left(\dfrac{dx}{x}\right)$ is $\tfrac{1}{2}x \cdot dx$, and its integral is $\tfrac{1}{4}x^2 + C$. Therefore equation (8) becomes:

$$\int x \log x \cdot dx = \tfrac{1}{2}x^2 \log x - \tfrac{1}{4}x^2 - C \tag{9}$$

For problems see group 8 in the appendix.

Special methods. Some differential expressions require more or less elaborate transformations or substitutions to get them into forms whose integrals can be recognized, or easily obtained by the simple transformations of rules III and IV above. Among these special methods are the following:

(a) Integration of rational fractions by resolution into partial fractions. *For problems illustrating this method see Group 9 in the Appendix.*

(b) Integration of expressions involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$ or $\sqrt{x^2 - a^2}$. Such expressions can usually be reduced to recognizable forms by making the following substitutions.

In $\sqrt{a^2 - x^2}$ substitute $a \sin \theta$ for x.

In $\sqrt{x^2 + a^2}$ substitute $a \tan \theta$ for x.

In $\sqrt{x^2 + a^2}$ substitute $a \sec \theta$ for x.

Problems illustrating these trigonometric substitutions are given in Group 10 of the Appendix.

(c) Miscellaneous Substitutions. *A few problems involving miscellaneous substitutions are given in Group 11 in the Appendix.*

A good discussion of integration transformations is given on pages 32–70 of W. E. Byerly's *Integral Calculus*, Ginn & Co., Boston, 1881; second edition, 1888.

CHAPTER IV.

PARTIAL DIFFERENTIATION AND INTEGRATION.

58. Differentiation of a function of two variables.—Consider a rectangle of length x and breadth y as shown in Fig. 40. The area of the rectangle is:

$$A = xy \tag{1}$$

that is, the area is a function of the two variables x and y.

Writing $A + \Delta A$ for A,
$x + \Delta x$ for x, and
$y + \Delta y$ for y, we have:

$$A + \Delta A = (x + \Delta x)(y + \Delta y)$$

or

$$A + \Delta A = xy + x \cdot \Delta y + y \cdot \Delta x + \Delta x \cdot \Delta y \tag{2}$$

whence, subtracting equation (1) from equation (2) member by member, we have

$$\Delta A = x \cdot \Delta y + y \cdot \Delta x + \Delta x \cdot \Delta y \tag{3}$$

But when Δx and Δy are infinitely small we may drop the second order infinitesimal $\Delta x \cdot \Delta y$, and we get:

$$dA = x \cdot dy + y \cdot dx \tag{4}$$

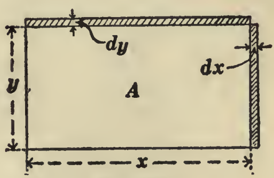


Fig. 40.

Now $x \cdot dy$ is the infinitesimal increment of A when y alone increases, and $y \cdot dx$ is the infinitesimal increment of A when x alone increases. Therefore, according to equation (4), *the infinitesimal increment of A when x and y both increase is equal to the sum of the two infinitesimal increments:—(a) that which is due to the increase of x alone and (b) that which is due to the increase of y alone.*

This proposition is a special case of a general theorem which is expressed by the equation:

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy \tag{5}$$

where $\frac{\partial z}{\partial x}$ is the derivative of z with respect to x *on the assumption that y is constant*, and $\frac{\partial z}{\partial y}$ is the derivative of z with respect to y *on the assumption that x is constant*; that is $\frac{\partial z}{\partial x} \cdot dx$ is the infinitesimal increment of z due to an increment of x alone, $\frac{\partial z}{\partial y} \cdot dy$ is the infinitesimal increment of z due to an increment of y alone, and $dz\left(=\frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy\right)$ is the infinitesimal increment of z when x and y both increase.

It is customary to use the symbol ∂ instead of d in partial differentiation, but to do so is apt to be misleading because it gives the impression that there is an unexplained and mysterious difference between ordinary and partial differentiation. It is indeed allowable to use d instead of ∂ because one always *knows* when one is dealing with a function of more than one independent variable.

59. Successive partial differentiation.—Let y be a function of a single independent variable x. Then the rate of change of y with respect to x is called the derivative of y with respect to x. But this derivative is itself a function of x and its rate of change with respect to x is called the second derivative of y with respect to x; and so on as explained in Art. 46.

Let z be a function of two independent variables x and y. Then the two first derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, and each of these first derivatives has its derivative with respect to x and its derivative with respect to y. For

example, if $z = x^3y^3$, then $\frac{\partial z}{\partial x} = 3x^2y^3(=\alpha)$, and $\frac{\partial z}{\partial y} = 3x^3y^2(=\beta)$; and each of these first derivatives is a function of x and y. The derivative $\frac{\partial z}{\partial x}(=\alpha)$ has two derivatives, namely, $\frac{\partial \alpha}{\partial x}(=6xy^3)$ and $\frac{\partial \alpha}{\partial y}(=9x^2y^2)$; and the derivative $\frac{\partial z}{\partial y}(=\beta)$ has two derivatives, namely, $\frac{\partial \beta}{\partial x}(=9x^2y^2)$ and $\frac{\partial \beta}{\partial y}(=6x^3y)$.

The derivative of $\frac{\partial z}{\partial x}$ with respect to x is indicated as $\frac{\partial^2 z}{\partial x^2}$.

The derivative of $\frac{\partial z}{\partial x}$ with respect to y is indicated as $\frac{\partial^2 z}{\partial y \cdot \partial x}$.

The derivative of $\frac{\partial z}{\partial y}$ with respect to x is indicated as $\frac{\partial^2 z}{\partial x \cdot \partial y}$.

The derivative of $\frac{\partial z}{\partial y}$ with respect to y is indicated as $\frac{\partial^2 z}{\partial y^2}$.

In the above example, namely, when $z = x^3y^3$, the two second derivatives, $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$, are each equal to $9x^2y^2$. That is these two second derivatives are identical. *Indeed the two derivatives* $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ *are always identical when* z *is a function of* x *and* y. This proposition is very important in mathematical physics.

60. Partial differential equation.*—Let z be a function of x and y concerning which it is known that

$$\frac{\partial z}{\partial x} = ax^2y \tag{1}$$

and let it be required to find an expression for z. Now y is assumed to be a constant when the derivative $\frac{\partial z}{\partial x}$ is found from a

* A very interesting example of a partial differential equation is discussed in Art. 90.

given function, and therefore y is to be looked upon as a constant in equation (1) when one tries to think of the function from which equation (1) is derived, that is, ay in equation (1) is to be considered as a constant, so that we may write

$$ay = b \tag{2}$$

and consider what function of x has a derivative equal to bx^2. Evidently the desired function of x is $[\frac{1}{3}bx^3 + C]$. But in this argument y is assumed to be constant, so that the constant of integration C can be any function whatever of y. Therefore, writing $f(y)$ for C and using ay for b, we have $[\frac{1}{3}ayx^3 + f(y)]$ as the most general algebraic expression which gives ax^2y when differentiated with respect to x, and this is therefore the desired expression for z. That is:

$$z = \tfrac{1}{3}ax^3y + f(y) \tag{3}$$

where $f(y)$ represents any function of y whatever. Of course this function $f(y)$ may include a term which does not contain y, as in $ay^2 + by + c$, or indeed it may be a simple constant which does not contain y at all.

Equation (1) expresses the law of growth of z with respect to x, and it is called a *partial differential equation* because z is a function of more than one independent variable.

Example of an ordinary differential equation.

If

$$\frac{dz}{dx} = ax^2 \tag{4}$$

then

$$z = \tfrac{1}{3}ax^3 + C \tag{5}$$

where C is a constant which may have any value whatever.

If we place $x = 0$ in equation

Example of a partial differential equation.

If

$$\frac{\partial z}{\partial x} = ax^2 \tag{6}$$

then

$$z = \tfrac{1}{3}ax^3 + f(y) \tag{7}$$

where $f(y)$ is any function whatever of y which does not depend on x.

(5) we have $z = C$. Therefore the constant C is determined if we *know* the value of z when $x = 0$.

The constant C is called *the constant of integration.*

If we place $x = 0$ in equation (7) we have $z = f(y)$. Therefore the function $f(y)$ is determined if we *know* z as a function of y when $x = 0$.

The function $f(y)$ is called *the function of integration.*

The above discussion refers to very simple partial differential equations. As another example let us set up the partial differential equation which characterizes a given function. Let z be any function whatever of s where s stands for $x + ay$. Then $\frac{ds}{dx} = 1$* and $\frac{ds}{dy} = a$. Also, according to Art. 34, we have:

$$\frac{dz}{dx} = \frac{dz}{ds} \cdot \frac{ds}{dx} \qquad \text{and} \qquad \frac{dz}{dy} = \frac{dz}{ds} \cdot \frac{ds}{dy}$$

Therefore, using the values, $\frac{ds}{dx} = 1$ and $\frac{ds}{dy} = a$, we have

$$\frac{dz}{dx} = \frac{dz}{ds} \qquad \text{and} \qquad \frac{dz}{dy} = a\frac{dz}{ds}$$

and consequently:

$$\frac{dz}{dy} = a\,\frac{dz}{dx} \tag{8}$$

This is, of course, a partial differential equation, and it is satisfied by any function whatever of $(x + ay)$. For example it is satisfied by $z = x + ay$, by $z = (x + ay)^2$, by $\sin(x + ay)$, by $\log(x + ay)$, by e^{x+ay}, etc.

61. Differentiation of an implicit function.—Any equation between x and y defines y as a function of x†. If the equation is solved for y, we have y as an *explicit function* of x; otherwise the equation defines y as an *implicit function* of x.

* In this discussion the symbol d is used instead of ∂ for partial differentiation.

† Or x as a function of y.

Thus

$$6x^2 + 10xy^3 + y^4 + 5 = 0 \qquad (1)$$

defines y as an implicit function of x. It is convenient to represent the entire left-hand member of equation (1) by the letter f. Imagine for a moment that f need not be equal to zero as required by equation (1), then $\frac{\partial f}{\partial x} \cdot dx$ would be the increment of f due to an increment of x, and $\frac{\partial f}{\partial y} \cdot dy$ would be the increment of f due to an increment of y. But according to equation (1) f is always equal to zero. Therefore the increments dx and dy must be so related to each other as to make the sum of the two increments of f equal to zero. That is we must have:—

$$\frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy = 0 \qquad (2)$$

or, solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad (3)$$

This formula gives the value of $\frac{dy}{dx}$ when y is an implicit function of x as defined by the equation $f = 0$, where f is any algebraic expression involving x and y.

In the above expression the two differentials dx and dy are called increments for the sake of brevity and clearness. A differential is of course not an increment because it is infinitely small.

PROBLEMS.

Using the method of Art. 61, find $\frac{dy}{dx}$ in the following expressions.

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$ $\qquad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

2. $x^3 + y^3 - 3axy - 3 = 0$, $\qquad \dfrac{dy}{dx} = \dfrac{x^2 - ay}{ax - y^2}$.

3. $6x^2 + 10xy^3 + y^4 + 5 = 0$, $\qquad \dfrac{dy}{dx} = -\dfrac{6x + 5y^3}{y^2(15x + 2y)}$.

4. $\log(x^2 - y^2) - 2\sin^{-1}\dfrac{y}{x} = 0$, $\qquad \dfrac{dy}{dx} = \dfrac{x^2 + y\sqrt{x^2 - y^2}}{x(y + \sqrt{x^2 - y^2})}$.

5. $2\tan^{-1}\dfrac{x}{y} - \log(x^2 + y^2) = 0$, $\qquad \dfrac{dy}{dx} = \dfrac{y - x}{y + x}$.

62. Slope of a hill.—Imagine a hill built upon the plane which contains the x and y axes of reference, and let z be the height of the hill above the point on the plane whose coördinates are x and y. Then an equation expressing z as a function of x and y is the equation of the surface of the hill. Thus

$$z = \sqrt{r^2 - x^2 - y^2} \tag{1}$$

is the equation of a hemispherical hill of radius r.

Consider the derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$; these derivatives are both functions of x and y. Thus if $z = \sqrt{r^2 - x^2 - y^2}$, we have

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{r^2 - x^2 - y^2}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{r^2 - x^2 - y^2}}.$$

Let x', y' and z' be the coördinates of a particular point on the surface of the hill as shown in Fig. 41. If the values x' and y' are substituted for x and y in the general expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ *we get the values of these derivatives at the point* p, and we will represent these values by $\left(\dfrac{\partial z}{\partial x}\right)_p$ and $\left(\dfrac{\partial z}{\partial y}\right)_p$. Then $\left(\dfrac{\partial z}{\partial x}\right)_p$

is the slope of the tangent line qq and it is equal to $\tan\alpha$ where α is the angle shown in Figs. 41a and 41b; and $\left(\dfrac{\partial z}{\partial y}\right)_p$ is the slope

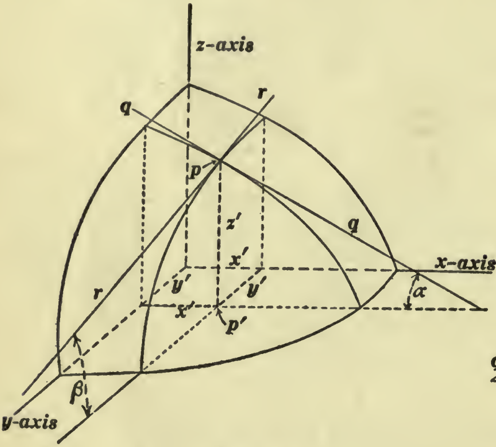


Fig. 41a.

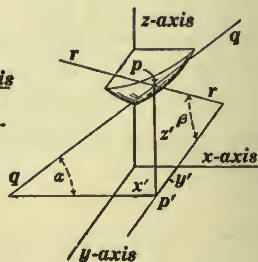


Fig. 41b.

of the tangent line rr and it is equal to $\tan\beta$ where β is the angle shown in Figs. 41a and 41b.

In Fig. 41a $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are both negative; that is, z decreases as x increases, and z decreases as y increases. In Fig. 41b $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are both positive.

63. Equation of the tangent plane.—The derivation of the equation of the tangent plane which touches a surface at a point is somewhat similar to the derivation of the equation of a tangent line which touches a curve at a point. Therefore it is worth while to give the derivation of the equation of a tangent line before considering the equation of a tangent plane.

Let x' and z' be the coördinates of the point p where the tangent line touches the plane curve cc, as shown in Fig. 42a.

Starting at the point p in Fig. 42a, which is at a height z' above the base line, travel along the tangent line tt covering the hori-

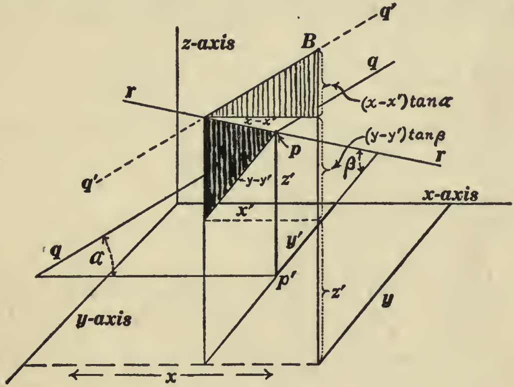


Fig. 42b.

zontal distance $(x - x')$ and the rise will be $(x - x')\tan\alpha$ or $(x - x')\,\dfrac{dz}{dx}$. We thus reach any given point B in the tangent line of which the coördinates are x and z, and the value of z is

$$z = z' + (x - x')\,\frac{dz}{dx} \tag{1}$$

which is the desired equation of the tangent line tt.

Let x', y' and z' be the coördinates of the point p where the tangent plane touches the given surface (see Fig. 42b). Starting at the point p, which is at a distance z' above the base plane, travel along the tangent line rr (which lies in the tangent plane) covering the horizontal distance $(y - y')$ towards the reader in Fig. 42b, and the rise will be

$$(y - y')\tan\beta = (y - y')\left(\frac{\partial z}{\partial y}\right)_p$$

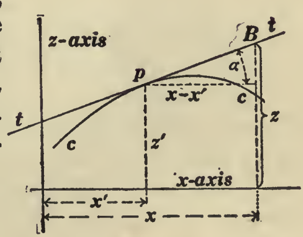


Fig. 42a.

Then continue along the line $q'q'$ parallel to the tangent line qq (the line $q'q'$ lies in the tangent plane) covering the horizontal distance $(x - x')$ to the right in Fig. 42*b*, and the rise will be

$$(x - x')\tan\alpha = (x - x')\left(\frac{\partial z}{\partial x}\right)_p$$

We thus reach any given point B in the tangent plane of which the coördinates are x, y and z, and the value of z is:

$$z = z' + (y - y')\left(\frac{\partial z}{\partial y}\right)_p + (x - x')\left(\frac{\partial z}{\partial x}\right)_p \tag{2}$$

which is the desired equation of the tangent plane. Of course the derivatives in this equation refer to the equation of the hill.

64. Component slopes and resultant slope.—Let X be the slope of a hill at the point p, Fig. 41, in a direction parallel to the x-axis. That is, X is the rise* per unit-horizontal-distance-parallel-to-the-x-axis, and it is equal to $\left(\frac{\partial z}{\partial x}\right)_p$. Also let Y be the slope of the hill at p in a direction parallel to the y-axis. Then $Y = \left(\frac{\partial z}{\partial y}\right)_p$. Let X and Y be represented to a chosen scale by the lines X and Y in Fig. 43. *Then the diagonal R in Fig. 43 represents what may be called the resultant or actual slope of the hill at p.* That is, the line R points directly up hill, and the length of R represents (to the chosen scale) the rise of the hill per unit-of-horizontal-distance-in-the-direction-of-R. This is a fundamental and important theorem in the mathematical theory of electricity and magnetism, and the proof of the theorem is as follows:

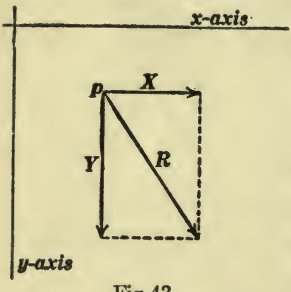


Fig 43.

The x and y axes are shown as they would appear if seen from above in Figs. 41*a* and 41*b*.

* If the slope is negative, X is the drop per unit horizontal distance, etc.

Let the line AB in Fig. 44 be the line of intersection of a horizontal plane through p with the plane which is tangent to the hill at p. Then the line AB is evidently at right angles to

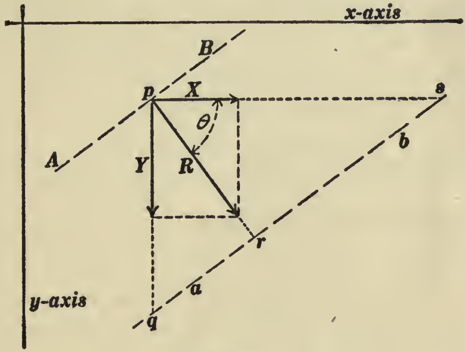


Fig. 44.

R. Let the line ab *be the line in the tangent plane at every point of which the tangent plane is unit distance above the horizontal plane through* p. Then, by definition:

$$R = \frac{1}{\overline{pr}} \tag{1}$$

$$X = \frac{1}{\overline{ps}} \tag{2}$$

and

$$Y = \frac{1}{\overline{pq}} \tag{3}$$

where $\overline{pr}$, $\overline{ps}$ and $\overline{pq}$ are the distances shown in Fig. 44. But $\overline{ps} = \frac{\overline{pr}}{\cos\theta} = \frac{1}{R\cdot\cos\theta}$, and $\overline{pq} = \frac{\overline{pr}}{\sin\theta} = \frac{1}{R\cdot\sin\theta}$. Therefore

$$X = R\cos\theta \tag{4}$$

$$Y = R\sin\theta \tag{5}$$

and

$$X^2 + Y^2 = R^2 \tag{6}$$

These equations establish the proposition that R is the diagonal of a rectangle of which X and Y are the sides as shown in Fig. 44.

65. Example of gradient in two dimensions.—The observed pull in Fig. 10 is a function of one independent variable, the current as indicated by the ammeter, and in such a case it is helpful to plot a *curve* of which the abscissas represent observed currents and of which the corresponding ordinates represent observed values of pull. This curve is shown in Fig. 11.

Frequently one is concerned with a quantity which depends upon two independent variables. For example one might be concerned with the distribution of temperature over a flat metal plate. In such a case it is helpful to think of the temperature at each point p as represented by the height at p of a *hill*. Then the component gradients of temperature are represented everywhere by the component slopes of the hill, and the resultant gradient of temperature is represented everywhere by the resultant slope of the hill, as indicated by the sides and diagonal of the rectangle in Fig. 45.

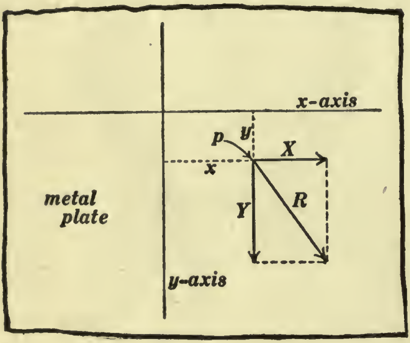


Fig 45.

66. Example of gradient in three dimensions.—One might be concerned with the distribution of temperature throughout a solid body. Of course, *the temperature would have a definite value at each point in a body*, or, as expressed in mathematical language, the temperature T at each point would be a definite function of the coördinates x, y and z of the point, and the derivatives $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$ would also be functions of x, y and z. That is, the

three derivatives would have definite values at each point in the body, and the three derivatives, $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$, are the component gradients of the temperature at the point in directions parallel to the respective axes of reference. Representing the resultant temperature gradient at a point by R and the three components of R by X, Y and Z, we have

$$R^2 = X^2 + Y^2 + Z^2 \tag{1}$$

$$X = \frac{\partial T}{\partial x} \tag{2}$$

$$Y = \frac{\partial T}{\partial y} \tag{3}$$

$$Z = \frac{\partial T}{\partial z} \tag{4}$$

These equations constitute an extension to three dimensions of the theorem of Art. 64.

In Fig. 45 the direction at right angles to the surface of the plate is available for purposes of geometrical representation, and one can think of an actual hill being built upon the plate so that the temperature of the plate at each point is represented by the height of the hill at that point; and of course this "hill" becomes a "valley" dipping below the plate where the temperature of the plate is below zero.

The temperature at each point of a solid body, however, cannot be represented geometrically as a height because space is filled in every direction by the body itself. That is, every direction in space is used for representing the independent variables x, y and z, and no direction is left for the representation of T. It is convenient, however, even in this case, to speak of the "*temperature hill*," the "*height*" of the hill at each point of the body being the *temperature itself*, high or low as the case may be.

67. Partial integrations.*—In many cases the derivative of a

* What is here called partial integration is usually called *multiple integration*, double, triple, quadruple, etc., as the case may be.

function of one independent variable can be set up or established by the use of elementary principles of physics and arithmetic, and the function itself can be found by one integration, as exemplified in Arts. 22 and 23.

But when the function to be established is a function, say, of two independent variables (x and y for example), then it is usually necessary to perform an integration with respect to x in setting up the derivative of the desired function with respect to y, or to perform an integration with respect to y in setting up the derivative of the desired function with respect to x. Such integrations may be called *partial integrations* because they are related to partial differentiation in the same way that ordinary integration (with respect to one independent variable) is related to ordinary differentiation (with respect to one independent variable).

68. Volume of a hill. Example of partial integrations.—For the sake of simplicity let us consider a particular case, namely, a hemispherical hill with its center at the origin of coördinates. Then the equation of the surface of the hill is

$$z = \sqrt{r^2 - x^2 - y^2} \tag{1}$$

Thus Fig. 41a represents one quarter of a hemispherical hill, and it is required to find its volume.

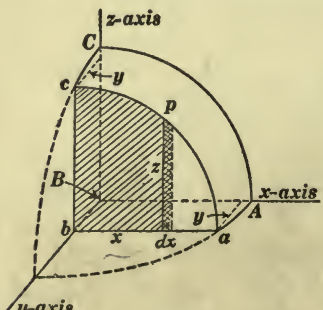


Fig. 46a.

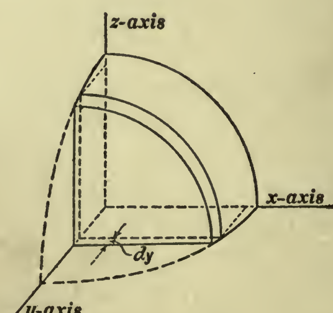


Fig. 46b.

Let v be the volume of the portion $abcABC$ of the hill, as shown in Fig. 46a. Then v is a function of y (see figure), and the increment of v due to an infinitesimal increment of y is the volume of the thin slab in Fig. 46b. Therefore we have:

$$dv = A \cdot dy \tag{2}$$

where A is the area of the face of the slab as shown in Fig. 46b. Therefore we must find an expression for A before we have a known expression for $\frac{dv}{dy}$.

Now apc in Fig. 46a is a plane curve, and equation (1) is the equation of this curve (ordinate z, abscissa x) if y is constant. Let α be the shaded area in Fig. 46a; then the strip with double shading is $d\alpha$ and it is equal to $z \cdot dx$. Therefore using the value of z from equation (1) we have:

$$d\alpha = \sqrt{r^2 - x^2 - y^2} \cdot dx \tag{3}$$

and the total area A of the face $abcp$ is the integral of this expression from $x = 0$ to $x = ba = \sqrt{r^2 - y^2}$.* Therefore:

$$A = \int_{x=0}^{x=\sqrt{r^2-y^2}} \sqrt{r^2 - x^2 - y^2} \cdot dx \tag{4}$$

In performing this integration y is a constant. This integral gives an expression for A, and the value of A so found can be substituted in equation (2). Then equation (2) may be integrated between the limits $y = 0$ to $y = r$ to give the desired expression for the volume V of the hill. That is

$$V = \int_{y=0}^{y=r} A \cdot dy \tag{5}$$

The above discussion applies to a particular case, namely, to a quarter of a hemispherical hill; but the volume of any hill what-

* This value of ba is the value of x (for the given constant value of y) when $z = 0$, as found from equation (1).

ever can be found in the same way. Find by a first integration, the area A of a plane section of the hill parallel to the xz plane and at a distance y therefrom, and then integrate $A \cdot dy$ to get the required volume; or find by a first integration the area A' of a plane section of the hill parallel to the yz plane and at a distance x therefrom, and then integrate $A' \cdot dx$ to get the required volume.

The prismoid formula.—A prismoid is a solid with *parallel* top and bottom faces with sides generated by straight lines. Thus a cylinder is a prismoid, any wedge even if its base is a polygon or curve is a prismoid, Fig. 47*a* represents a prismoid.

The volume of a prismoid is given by the formula

$$V = \frac{T + 4M + B}{6} \times h \tag{6}$$

where T is the area of the top face, B is the area of the bottom face, M is the area of the middle section (which is parallel to top and bottom faces) and h is the altitude of the prismoid.

Equation (6) also gives the volume of any figure bounded by a second degree surface ss in Fig. 47*b* between parallel top and

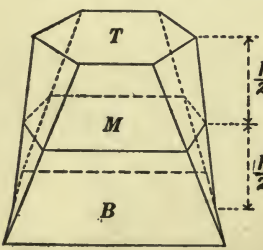


Fig. 47*a*.

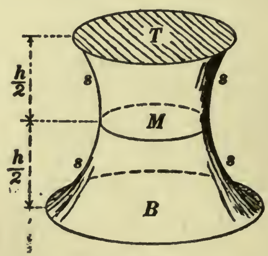


Fig. 47*b*.

bottom faces, and equation (6) can be used to give a very close approximation to the volume of any solid between parallel ends with sides smoothly curved from end to end.

PROBLEMS.

1. Integrate equations (4) and (5), above, and find the volume of one quarter of a hemispherical hill as shown in Fig. 41*a*. Verify the result from the formula: volume of a sphere equals $\frac{1}{6}\pi$ times its diameter cubed.

2. Find the volume of the entire wedge (20 inches long) in Fig. *p*2, the wedge being 8 inches wide in a direction perpendicular to the plane of the paper.

Ans. 800 cubic inches.

Note.—The volume of a wedge is, of course, easily found by using the principles of elementary geometry. It is intended, however, that this problem be solved by integration as follows: Let v be the volume of the shaded portion of the wedge as shown in Fig. *p*2. Then v is a function of x, and the increment of v due to an increment of x is the volume of the thin slab shown in Fig. *p*2. The area of the face of the slab is $\frac{x}{20} \times 10$ inches $\times 8$ inches $= 4x$ square inches. Therefore the volume of the slab is $4x \cdot dx$ cubic inches. That is,

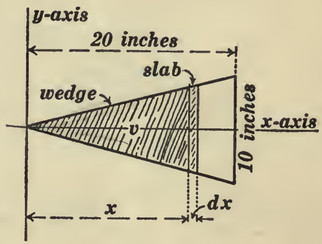


Fig. *p*2.

$$dv = 4x \cdot dx$$

and the desired volume is found by integrating this expression from $x = 0$ to $x = 20$ inches.

3. Find the volume of the shaded portion of the wedge in

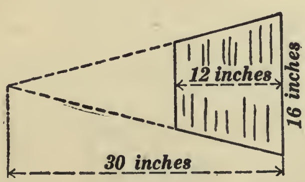


Fig. *p*3.

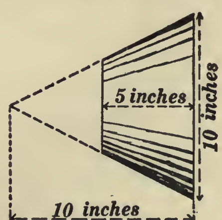


Fig. *p*4.

Fig. $p3$, the wedge being 6 inches wide in a direction perpendicular to the plane of the paper.

Ans. 921.6 cubic inches.

4. Find the volume of the frustum of a cone which is shown in Fig. $p4$.

Ans. 229.2 cubic inches.

5. Find the volume of the paraboloid of revolution which is shown in Fig. $p5$.

Ans. 1060.7 cubic inches.

Note.—The equation of the paraboloid is $y = px^2$, where the axis of revolution is the y-axis of reference; and the value of p is determined from the condition that $x = 7\frac{1}{2}$ inches when $y = 12$ inches.

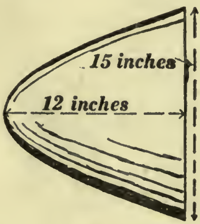


Fig. $p5$.

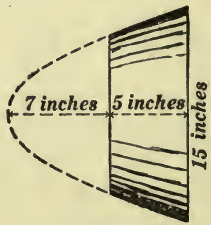


Fig. $p6$.

6. Find the volume of the shaded portion of the paraboloid in Fig. $p6$.

Ans. 699.8 cubic inches.

7. Find the volume of the spherical segment which is shown in Fig. $p7$.

Ans. 276.79 cubic inches.

8. Find the volume of the segment of a sphere which is shown in Fig. $p8$.

Ans. 119.71 cubic inches.

9. Find the volume between the xy plane and the plane $z = a + bx + cy$ and inside of the cylinder $(x - e)^2 + (y - f)^2 = g^2$ when $a = 10$ inches, $b = 3$, $c = 2$, $e = 8$ inches, $f = 9$ inches and $g = 5$ inches.

Ans. 4085.7 cubic inches.

Note.—Let $A \cdot dy$ be the expression which must be integrated with respect to y to give the desired volume. Then A is obtained by integrating $z \cdot dx$ between limits which are the two values of x found from $(x-e)^2 + (y-f)^2 = g^2$ for the given value of y, and then $A \cdot dy$ is integrated between the limits $y = f + g$ and $y = f - g$.

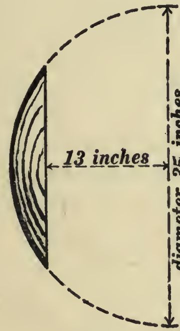


Fig. *p*7.

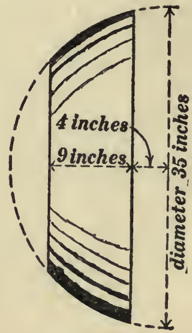


Fig. *p*8.

10. Find the total volume which is common to the two cylinders: $y^2 + z^2 = 25$ and $x^2 + y^2 = 25$, everything being expressed in inches.

Ans. 666.7 cubic inches.

Note.—One should be able to determine the proper limits for each integration in this problem if one understands problem 9.

11. The inside dimensions of a barrel are 20 inches in diameter at each end, 23 inches in diameter at the middle and 31 inches long. What is the capacity of the barrel in gallons?

Ans. 51.3 gallons.

69. Area of the surface of a hill. Another example of partial integration.—It is desired to find the area of the portion $caCA$ of the surface of the hill shown in Fig. 48 when the equation of the surface of the hill is given; this equation, of course, expresses the height z of the hill at a point as a function of x and y as explained in Art. 68. Let S be the area of $caCA$. Then the

increment of S due to an infinitesimal increment of y $(= Bb$ in the figure) is the area dS of the narrow strip ca. To find the area of this narrow strip let s be the area of the *portion* from c to ds in the figure, then s is a function of x, and the increment of s due to an infinitesimal increment of x is the small element of area ds which caps the prism $dx \,.\, dy$. This small element of area is sensibly coincident with the tangent plane at ds, and therefore the normal to ds makes an angle γ with the z-axis whose tangent is equal to

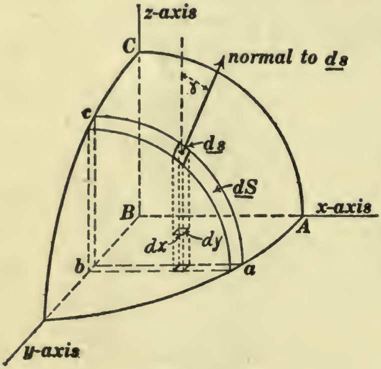


Fig. 48.

$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ which is the resultant slope of the hill at ds, according to Art. 64.

Now the area of the base of the prism, $dx \cdot dy$, is the projection of the area ds, and consequently we have:

$$dx \cdot dy = \cos\gamma \cdot ds$$

or

$$ds = \frac{dx \cdot dy}{\cos\gamma} \tag{1}$$

But if

$$\tan\gamma = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

we find:

$$\cos\gamma = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \tag{2}$$

so that equation (1) becomes:

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \cdot dx \cdot dy \tag{3}$$

Now the expression under the radical is a function of x and y which may be found by differentiating the equation of the surface of the hill (which is given); and the total area of the strip ca (which is equal to dS) is the integral of (3) from $x = 0$ to $x = ba$* In this integration y and dy are constants. The value of dS so found is the required infinitesimal increment of the area $caCA$ due to an infinitesimal increment of y, and the value of the area $caCA$ is found by integrating this expression for dS between the limits $y = 0$ and $y = y$ (meaning any value of y). If the entire area of the hill in Fig. 48 is to be found, the expression for dS must be integrated from $y = 0$ to $y =$ the intercept of the hill on the y-axis.†

PROBLEMS.

1. The hill shown in Fig. 48 is a quarter of a hemisphere. Find the area of its surface and verify your result from the formula: area of a sphere $= 4\pi$ times its radius squared.

2. Find the area of the curved surface of the cone shown in Fig. $p2$.

Ans. 135.8 square inches.

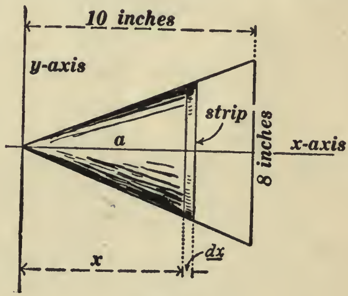


Fig. $p2$.

Note.—Let a be the area of the shaded portion of th cone in Fig. $p2$. Then a is a function of x; and the increment of a due to an increment of x is the area of the narrow strip shown in the figure. The width of this strip (parallel to the slant height of the cone) is $\dfrac{\sqrt{10^2 + 4^2}}{10} \times dx$, and the length of the strip (circumference of the cone) is $\dfrac{x}{10} \times 8 \times \pi$ inches. Therefore we get:

$$da = 0.864\pi x \cdot dx$$

* The value of ba is found from the given equation of the surface of the hill. It is the value of x for the given constant value of y when $z = 0$.

† This intercept is found from the equation of the surface of the hill by placing x and z both equal to zero and solving for y.

and the required area is found by integrating this expression from $x = 0$ to $x = 10$ inches.

3. Find the area of the curved surface of the frustum of the cone in Fig. $p3$.

Ans. 144.38 square inches.

4. Find the area of the part of the surface of the cylinder

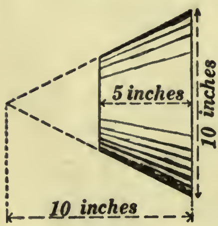


Fig. $p3$.

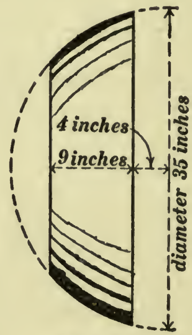


Fig. $p5$.

$x^2 + z^2 = 100$ which is inside of the cylinder $x^2 + y^2 = 100$, the radii of the cylinders being measured in inches.

Ans. 800 square inches.

5. Find the area of the convex surface of the segment of a sphere which is shown in Fig. $p5$.

Ans. 989.5 square inches.

6. Find the area of a zone on the earth's surface between 40° and 80° north latitude, taking the earth as a sphere 4,000 miles in radius.

Ans. 34,500,000 square miles.

CHAPTER V.

MISCELLANEOUS APPLICATIONS.

70. Influence of errors of observation upon a result. When one measures a thing with great care repeatedly the successive measurements never agree with each other, and it is easy to determine what is called the *probable error** of the measurement by considering the discrepancies in a set of observed values. Thus the length of a bar might be found by repeated measurement to be 103.625 centimeters with a probable error of, say, $\pm$ 0.0036 centimeter; the mass of a body, as determined by repeated weighing on a balance, might be 67.2382 grams with a probable error of, say, $\pm$ 0.0006 gram.

In many cases, however, it is not the directly measured quantities that are important but a *result* which is calculated therefrom; and it is often desirable to calculate the probable error in such a result when the probable errors in the individual measurements are known. For the sake of simplicity we will discuss a special case, as follows:

A steel ball is weighed on a balance and its mass is found to be M grams with a probable error of $\pm m$ grams, and the diameter of the ball is measured and found to be L centimeters with a probable error of $\pm l$ centimeters. The density of the steel as calculated from the measured values of M and L is:

$$D = \frac{M}{\frac{1}{6}\pi L^3} \tag{1}$$

and it is required to find the probable error in D. The probable

*See Franklin, Crawford and MacNutt's *Practical Physics*, Vol. I, pages 1–9 for a very simple discussion of errors of observation. For more complete discussion see Merriman's *Least Squares*, John Wiley & Sons, New York. See also Palmer's *Theory of Measurements*, McGraw-Hill Co., New York, 1912.

errors m and l are usually very small so that it is sufficiently accurate to calculate ΔD_m (the probable error in D due to $\pm m$) and ΔD_l (the probable error in D due to $\pm l$) by the formulas:

$$\Delta D_m = \frac{\partial D}{\partial M}\cdot m \tag{2}$$

and

$$\Delta D_l = \frac{\partial D}{\partial L}\cdot l \tag{3}$$

Let ΔD be the probable error in the result due to m and l together. Then ΔD is not equal to the sum $\Delta D_m + \Delta D_l$, but according to the theory of probability we have:

$$\Delta D = \sqrt{(\Delta D_m)^2 + (\Delta D_l)^2} \tag{4}$$

PROBLEMS.

1. The scale of an alternating current ammeter reads in degrees, and the value of the current is $i = k\sqrt{\theta}$, where k is a constant and θ is the deflection in degrees due to a current of i amperes. The error of reading is probably $\pm \frac{1}{4}$ degree. What is the probable error in the result, i, in fractions of an ampere (a) when the deflection is in the neighborhood of 200° and (b) when the deflection is in the neighborhood of 20°? Take the value of k to be such that 10 amperes gives 100° deflection. Ans: (a) $\pm$ 0.0089 ampere, (b) $\pm$ 0.028 ampere.

2. What is the percentage error (probable) in the value of i under conditions (a) and (b) in problem 1? Ans. (a) $\pm$ 0.06 per cent., (b) $\pm$ 0.6 per cent.

Note. The percentage error is $\frac{\Delta i}{i} \times 100$.

3. The diameter of a steel sphere is repeatedly measured and found to be 0.8762 inch with a probable error of $\pm$ 0.0016 inch. Calculate the volume of the sphere and calculate the probable error in the calculated volume. Ans. 0.3525 cubic inch $\pm$ 0.0019 cubic inch.

Note. It is interesting to note that the percentage error in the volume is three times as great as the percentage error in the measured diameter, that is $\frac{0.0019}{0.3525}$ is three times as great as $\frac{0.0016}{0.8762}$.

4. Show that the percentage error of x^2 is twice as great as the percentage error of x, when x is an observed quantity and x^2 is a calculated result.

5. By measurement the length of a rectangle is 25 inches with a probable error of $\pm$ 0.02 inch, and the width of the rectangle is 10 inches with a probable error of $\pm$ 0.015 inch. What is the probable error in the calculated area of 250 square inches? Ans. $\pm$ 0.421 square inch.

6. The diameter of a steel ball is 2.542 centimeters $\pm$ 0.001 centimeter, and the mass is 116.25 grams $\pm$ 0.02 gram. What is the density and what is the probable error in the density? Ans. 7.781 grams per cubic centimeter $\pm$ 0.0168 gram per cubic centimeter.

71. Maximum and minimum values. It is evident that a growing thing reaches its maximum size *when it stops growing;* but a thing may stop growing for a while and then go on growing

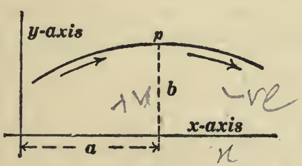


Fig. 49.

A maximum; $\frac{dy}{dx}$ is positive when x is less than a, and negative when x is greater than a.

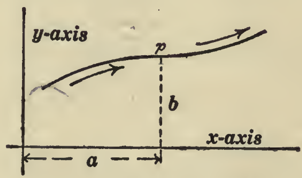


Fig. 50.

Not a maximum; $\frac{dy}{dx}$ is positive when x is less than a, also when x is greater than a.

again. Also it is evident that a decreasing thing reaches its minimum size *when it stops decreasing;* but a thing may stop decreasing for a while and then go on decreasing again. Thus

$\frac{dy}{dx}$ is equal to zero when $x = a$ in all four of the following figures, 49, 50, 51 and 52, but b is a maximum value of y in Fig. 49 only, and b is a minimum value of y in Fig. 51 only.

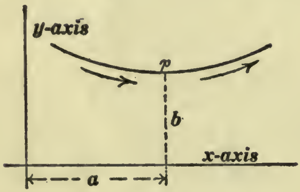


Fig. 51.

A minimum; $\frac{dy}{dx}$ is negative when x is less than a, and positive when x is greater than a.

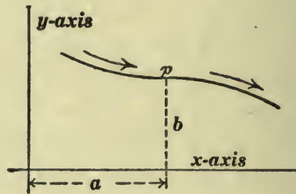


Fig. 52.

Not a minimum; $\frac{dy}{dx}$ is negative when x is less than a, also when x is greater than a.

To determine the values of x *for which a given function of* x *is a maximum or a minimum, differentiate the function, place its derivative equal to zero, and solve for* x. *Then for each value* a *of* x *so found, determine the algebraic sign of the derivative for a value of* x *slightly less than* a *and also for a value of* x *slightly greater than* a. *Interpret the results by referring to Figs. 49 to 52.**

PROBLEMS.

In the first five problems, following, find for what values of x is y a maximum or minimum and compute the maximum or minimum value in each case.

1. $y = x^3 - 6x^2 + 9x + 3.$ Ans. $x = 1$; $y = 7$, a maximum: $x = 3$; $y = 3$, a minimum.

2. $y = \dfrac{x^2 + 4x + 44}{x^2 + 8x + 40}.$ Ans. $x = 8$; $y = \frac{5}{6}$, a minimum: $x = -6$; $y = 2$, a maximum.

* A slightly easier method is to consider the values of the second derivative of the given function but the above method is sufficient.

3. $y = (x-4)^3(x+1)^2$. Ans. $x = -1$; $y = 0$, a maximum: $x = 1$; $y = -108$, a minimum.

4. $y = 9e^{2x} + 25e^{-2x}$. Ans. $x = \log \sqrt{\frac{5}{3}}$; $y = 30$, a minimum

5. $y = 3 \sin x + 4 \cos x$. Ans. $x = \tan^{-1}\frac{3}{4}$ in the first quadrant; $y = 5$, a maximum: $x = \tan^{-1}\frac{3}{4}$ in the third quadrant; $y = -5$, a minimum.

6. Divide the number 20 into two parts such that the product of one part by the square of the other part shall be a maximum. Ans. 13.33 and 6.67.

7. What number added to one half the square of its reciprocal gives the least possible sum. Ans. 1.

8. The cost of an electrical power transmission line is approximately proportional to the weight of copper used because the cost of poles and the cost of erection can be expressed with a fair degree of accuracy as an addition of so many cents per pound on the cost of copper. Therefore the annual money cost represented by interest on investment and depreciation, being reckoned as so many per cent. of first cost, is proportional to the weight of copper used, or it is equal to kw where k is a constant and w is the weight of copper used.

The amount of power lost in a transmission line is halved if the amount of copper is doubled. Therefore the annual money loss due to loss of energy in the line is universely proportional to the weight of copper, or it is equal to $\frac{m}{w}$ where m is a constant.

Therefore the total annual money loss is $kw + \frac{m}{w}$. Find the value of w (expressed in terms of k and m) which will make this total annual money loss a minimum. Ans. $w = \sqrt{\frac{m}{k}}$

Note. This matter is discussed in electrical engineering books under the head of Kelvin's law. See Franklin's *Electric Lighting*, Art. 26, pages 66–70. The Macmillan Co., New York, 1912.

8. It is desired to construct of sheet metal a cylindrical gallon

measure without a cover. Find what its depth must be in order that the amount of sheet metal used shall be a minimum. Ans. $y = \sqrt[3]{\frac{231}{\pi}}$ (= 4.19) inches.

Note. The amount of sheet metal used will be a minimum when the surface area S of the vessel is a minimum; but $S = 2\pi xy + \pi x^2$, where x is the radius of the base of the cylinder and y is its depth. Since the measure is to contain 1 gallon (= 231 cubic inches), $\pi x^2 y = 231$; and, substituting for x its value in terms of y, we have:

$$S = 2\sqrt{231\pi y} + \frac{231}{y}$$

The value of y which makes this expression a minimum is to be obtained.

10. Determine the altitude of the cylinder of greatest convex surface that can be inscribed in a sphere whose radius is 10 inches. Ans. 14.142 inches.

11. Determine the altitude of the cylinder of greatest volume that can be inscribed in a sphere whose radius is 10 inches. Ans. 11.547 inches.

12. Determine the volume of the smallest cone which can be circumscribed about a sphere whose radius is 10 inches. Ans. $\frac{800\,\pi}{3}$ (= 838.1) cubic inches.

13. A man who can row at a speed of 4 miles per hour and run at a speed of 6 miles per hour wishes to reach the point p from a boat at b as shown in Fig. $p13$ in the least possible time. Find the distance ap that the man must run on the beach. Ans. 1.056 miles.

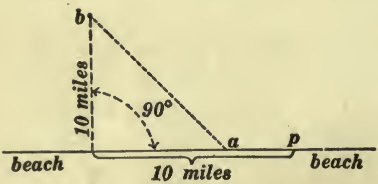


Fig. $p13$.

14. The strength of a rectangular beam is proportional to its breadth and to the square of its depth. Find the breadth and depth of the strongest beam which can be cut from a cylindrical

log whose radius is 18 inches. Ans. Breadth, 20.8 inches; depth, 29.4 inches.

15. For what angle of deflection does the old-fashioned tangent galvanometer give the greatest percentage accuracy in the value of the current? Ans. 45°.

Note. The equation of the tangent galvanometer is:

$$i = k \tan \theta$$

where i is the current in amperes, θ is the observed angle of deflection, and k is a constant. Let $d\theta$ represent the probable error of the observed value of θ, then the probable error in the value of i is:

$$di = \frac{k \, . d\theta}{\cos^2 \theta}$$

and the percentage error in the value of i is:

$$y = \frac{di}{i} = \frac{d\theta}{\sin \theta \cos \theta}$$

in which $d\theta$ being the probable error of θ is to be treated as a constant.

A general discussion of this important matter of precision as dependent upon the choice of magnitudes of the quantities to be observed is given in Chapter XII of Palmer's *Theory of Measurements*, McGraw-Hill Co., New York, 1912.

72. The problem of the bent beam. The formulas used in connection with a bent beam are based on Hooke's law, and the

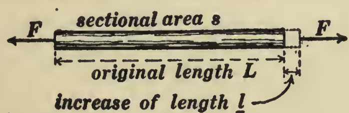


Fig. 53.

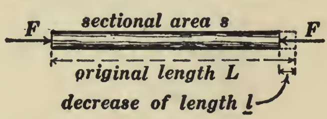


Fig. 54.

derivation of two of the simpler formulas furnishes a good example of the use of calculus.

Hooke's law. Figure 53 represents a rod subjected to a stretching force F. Experiment shows that $\frac{F}{s}$ is proportional

to $\frac{l}{L}$ (Hooke's law). Therefore we may write

$$\frac{F}{s} = E \cdot \frac{l}{L}$$

or

$$F = E \cdot \frac{ls}{L} \tag{1}$$

in which E is a proportionality factor and it is called the *stretch modulus* of the substance of which the rod is made. Equation (1) also gives the compressing force F required to produce a slight shortening of the rod as represented in Fig. 54.

When a beam is bent the filaments on one side of the beam are lengthened, and the filaments on the other side are shortened. If we wish to determine the degree of lengthening or shortening it is necessary to consider a very short portion of the beam. But a very short portion of a bent beam has the same curvature as the osculating circle as shown in Fig. 55, and *the easiest way*

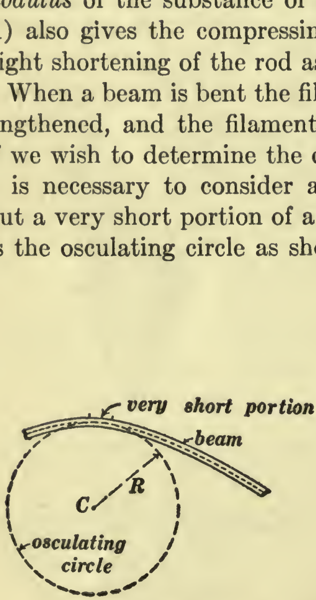


Fig. 55.

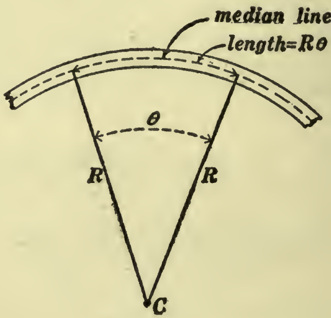


Fig. 56.

to think of the very short portion of the bent beam is to think of it as a portion of a long beam which is actually bent into the arc of a circle. Thus Fig. 56 represents a beam bent into the arc of a circle.

A certain filament in the bent beam is unchanged in length.

This filament is located in what is called the *median line* (or plane) of the beam. The median plane of a beam of rectangular section is the middle plane of the beam.

Let us consider the forces which act across a section qq of a bent beam as shown in Fig. 57. Considering these forces as acting on AB, they are a set of pulls on A and a set of pushes on B, and together they constitute a turning force or torque, T, about an axis O perpendicular to the plane of the paper. It is the object of this discussion to derive an expression for T in terms of b, D, R and E, where b is the breadth of the beam, D is the depth of the beam, R is the radius of curvature of the median line of the beam, and E is the stretch modulus of the material of which the beam is made.

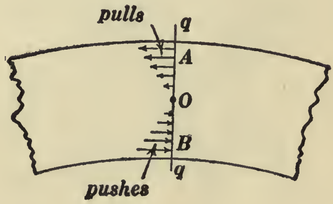


Fig. 57.

Let us consider the portion of the beam which lies between the radii RR in Fig. 56. This portion of the beam is shown to

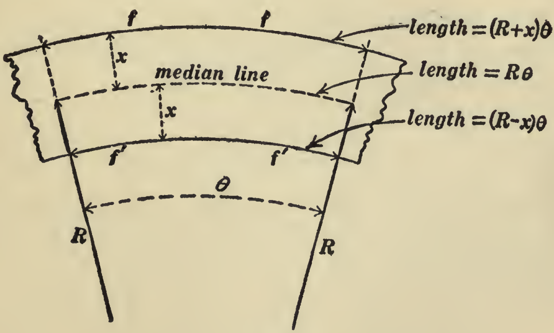
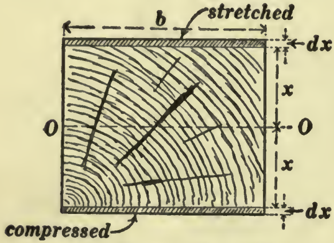


Fig. 58.

a larger scale in Fig. 58. The original length of every filament of this portion of the beam is $R\theta$. Consider the upper filament

ff of the beam. The stretched length of this filament is $(R + x)\theta$, because ff is a circular arc of radius $(R + x)$ and it subtends the angle θ. Therefore the filament ff has been increased in length by the amount $x\theta$ by the bending. Similarly it may be shown that the lower filament $f'f'$ in Fig. 58 has been shortened by the amount $x\theta$ by the bending.



sectional view of beam

Fig. 59.

Figure 59 is a sectional view of the beam with its depth arbitrarily increased by added layers of material on top and bottom, and we wish to find the increment of T due to this assumed increase of depth of the beam. The stretching force in the top layer can be found from equation (1) by substituting $x\theta$ for l, $b.dx$ for s and $R\theta$ for L, which gives

$$F = \frac{Eb}{R}\cdot x\cdot dx$$

this stretching force is a pull at A in Fig. 57, and its torque action about the axis OO in Fig. 59 is counter-clock wise as seen in Fig. 57, and it is equal to

$$Fx = \frac{Eb}{R}\cdot x^2 dx$$

The compressing force in the bottom layer is a push at B in Fig. 57, and its torque action about the axis OO is also counter-clock wise as seen in Fig. 57, and it is equal to

$$Fx = \frac{Eb}{R}\cdot x^2\cdot dx$$

Therefore the total torque action due to the two added layers is:

$$dT = \frac{2Eb}{R}\cdot x^2\cdot dx \tag{2}$$

Whence, by integrating, we have:

$$T = \frac{2}{3} \cdot \frac{Ebx^3}{R} + \text{a constant} \tag{3}$$

But it is evident that $T = 0$ when $x = 0$, because a beam of zero depth would of course have no stiffness at all. Therefore the constant of integration in equation (3) is zero, and equation (3) becomes:

$$T = \frac{2}{3} \frac{Ebx^3}{R} \tag{4}$$

It is somewhat simpler to express T in terms of the depth D of the beam $(= 2x)$. Therefore substituting $x = \frac{D}{2}$ in equation (4), we have:

$$T = \frac{1}{12} \frac{EbD^3}{R} \tag{5}$$

This equation applies to a beam bent in any way whatever, T being the torque action across a particular section of the beam, and R the radius of curvature of the median line at that point. Thus Fig. 60 shows a beam with one end fixed in a wall and carrying a weight W at the other end. In this case the bending torque at p is equal to $W(a - x)$, and the torque action across the section of the beam at p is equal and opposite to $W(a - x)$ Therefore, ignoring algebraic signs, we have

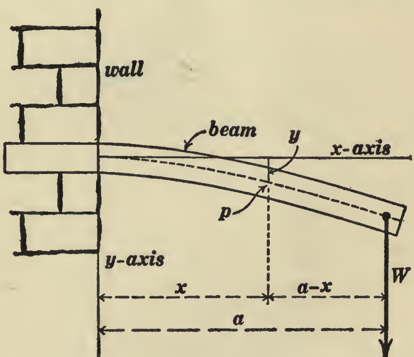


Fig. 60.

$$T = W(a - x) \tag{6}$$

Substituting this value for T in equation (5) and solving for R we have:

$$R = \frac{1}{12}\frac{EbD^3}{W(a-x)} \tag{7}$$

or

$$R = \frac{c}{a-x} \tag{8}$$

where

$$c = \frac{1}{12}\cdot\frac{EbD^3}{W} \tag{9}$$

Let it be required to find the equation of the curve formed by the median line of the beam in Fig. 60. According to Art. 49 the radius of curvature of any curve at a point is:

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \tag{10}$$

If the bending of the beam in Fig. 60 is very slight the value of $\frac{dy}{dx}$ will be everywhere very small so that, as a first approximation, we may neglect $\left(\frac{dy}{dx}\right)^2$ in the numerator of equation (10) where it is added to the much larger constant quantity 1. Therefore from equations (8) and (10) we have

$$\frac{1}{\frac{d^2y}{dx^2}} = \frac{c}{a-x} \tag{11}$$

or

$$\frac{d^2y}{dx^2} = \frac{a}{c} - \frac{x}{c} \tag{12}$$

whence, by integrating,* we have

$$\frac{dy}{dx} = \frac{ax}{c} - \frac{x^2}{2c} + \text{a constant} \tag{13}$$

* By recognizing the function of which $\frac{a}{c} - \frac{x}{c}$ is the derivative.

But $\dfrac{dy}{dx}$ is actually* equal to zero when $x = 0$ in Fig. 60, so that the constant of integration in equation (13) is zero, or:

$$\frac{dy}{dx} = \frac{ax}{c} - \frac{x^2}{2c} \tag{14}$$

Integrating this expression, we have:

$$y = \frac{ax^2}{2c} - \frac{x^3}{6c} + \text{a constant} \tag{15}$$

But $y = 0$ when $x = 0$ in Fig. 60, so that the constant of integration in equation (15) is zero, and equation (15) becomes:

$$y = \frac{ax^2}{2c} - \frac{x^3}{6c} \tag{16}$$

in which c the constant defined by equation (9).

73. Average value of a function. Let y be a function of x as represented by the curve cc in Fig. 61. The average value of y between $x = a$ and $x = b$ is the shaded area in Fig. 61 divided by the base $(b - a)$. This gives the height of a rectangle of base $(b - a)$ the area of the rectangle being equal to the shaded area in the figure. From this definition of average value we have:

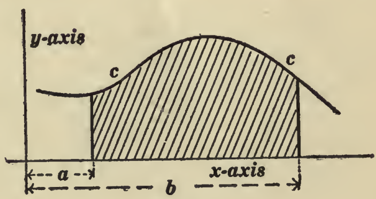


Fig. 61.

$$\left\{\begin{matrix}\text{average value of } y \\ \text{between } x = a \text{ and } x = b\end{matrix}\right\} = \frac{1}{b-a}\int_{x=a}^{x=b} y \cdot dx \tag{1}$$

PROBLEMS.

1. The velocity of a falling body is $v = gt$. Find the average value of v from $t = 0$ to $t = 10$. Ans. $5g$.

* Not approximately.

2. The velocity of a falling body, starting with an initial velocity u, is $v = u + gt$. Find the average value of v from $t = 5$ to $t = 10$. Ans. $u + 7.5\,g$.

3. Find the average value of $y = \sin x$ from $x = 0$ to $x = \pi$. The answer is shown in Fig. $p3$.

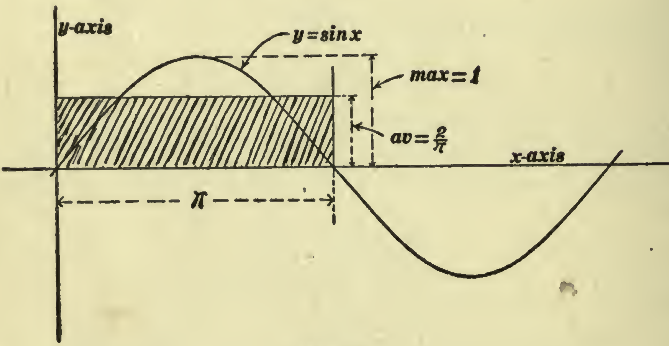


Fig. $p3$.

4. Find the average value of $e = E \sin \omega t$ during a whole cycle from $\omega t = 0$ to $\omega t = 2\pi$. Ans. Zero.

5. Find the average value of $y = \sin^2 x$ during a half cycle ($x = 0$ to $x = \pi$) and during a whole cycle ($x = 0$ to $x = 2\pi$). The answer is shown in Fig. $p5$.

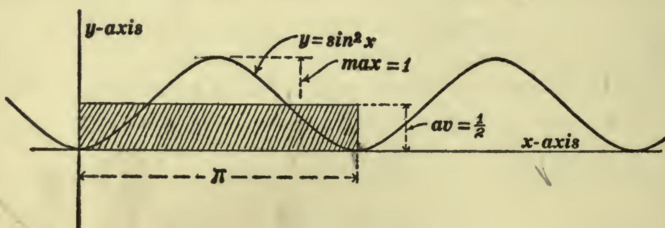


Fig. $p5$.

6. Find the average value of $y = \sin x \sin (x - \theta)$ during a whole cycle ($x = 0$ to $x = 2\pi$). Ans. $\frac{1}{2} \cos \theta$.

CENTER OF GRAVITY.

74. Resultant of a set of parallel forces. The single force R in Fig. 62 is equivalent to the three parallel forces A, B and C combined, and it is called the *resultant* of A, B and C. The *value* of R is equal to $A + B + C$, and the point of application of R is determined by the condition that the torque action of R about any arbitrarily chosen axis O (perpendicular to the plane of the paper) must be equal to the combined torque action of the given forces A, B, and C about that axis. That is:

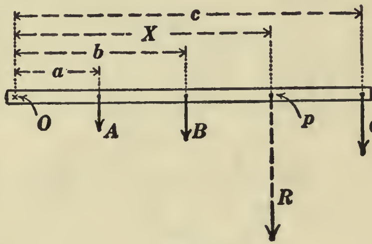


Fig. 62.

$$R = A + B + C \tag{1}$$

and

$$RX = Aa + Bb + Cc \tag{2}$$

so that

$$X = \frac{Aa + Bb + Cc}{A + B + C} \tag{3}$$

Definition of the center of gravity of a body. Every portion of a body is pulled downwards by gravity, and the forces which thus act upon the various parts of a body are together equivalent to a single force (their resultant) the point of application of which is called the *center of gravity* of the body. Thus the short arrows *fff* in Fig. 63 represent the forces with which gravity pulls on a body, these forces are together equivalent to the single force F, and the point of application of F is the center of gravity, C, of the body.

The force with which gravity pulls on a body is proportional to the mass, m, of the body. Therefore, if we use the pull of gravity on a one-pound body as our unit of force, the pull

of gravity on any body is numerically equal to the mass of the body in pounds. Thus a 10-pound body would be pulled by 10 units of force. This unit of force is called a *pound.* It is evident therefore that the word pound has two meanings; it signifies a unit of mass when we speak of so many pounds of sugar or coal, and it signifies a unit of force when we speak of so many pounds of force.

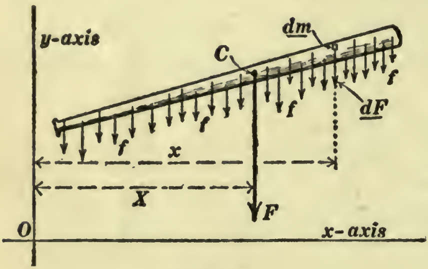


Fig. 63.

Consider a small particle of the body in Fig. 63 of which the mass is dm pounds and of which the abscissa is x as shown. The pull of gravity on the particle is $dF = dm,$ the torque action of this force about the axis O (perpendicular to the plane of the paper) is $x \cdot dm,$ the combined torque action about O of all the forces fff may be expressed as the integral $\int x \cdot dm,$ and this total torque action is equal to the torque action about O of the resultant F. Therefore we have:

$$FX = \int x \cdot dm \tag{4}$$

Now the force F is equal to the total mass M of the body as above explained. Therefore from equation (4) we have:

$$X = \frac{\int x \cdot dm}{M} \tag{5}$$

Similarly, the y and z coördinates of the center of gravity are given by the equations:

$$Y = \frac{\int y \cdot dm}{M} \tag{6}$$

and

$$Z = \frac{\int z \cdot dm}{M} \tag{7}$$

Equations (4) and (5) have exactly the same significance as equations (2) and (3); and when the value of $\int x \cdot dm$ is found for the whole body, then the distance X of the center of gravity from the origin O can be calculated if the total mass of the body is known.

75. Center of gravity of a bent rod. It is desired to find the coördinates X and Y of the center of gravity of a rod which is represented by the heavy black portion of the parabola which is, shown in Fig. 64 and whose equation is $y = kx^2$. Let h be

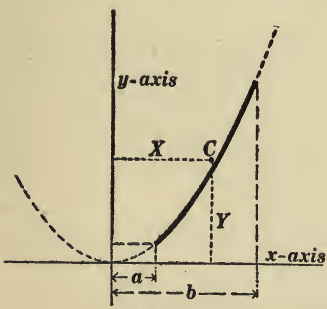


Fig. 64.

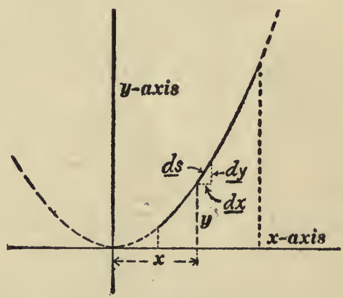


Fig. 65.

the mass of the rod per unit of length (pounds per foot). Then

$$h \cdot ds = h\sqrt{(dx)^2 + (dy)^2}$$

is the mass in pounds of the portion ds of the rod, as may be understood with the help of Fig. 65. But from the equation of the parabola, $y = kx^2$, we have

$$dy = 2kx \cdot dx.$$

Therefore $h\sqrt{(dx)^2 + (dy)^2}$ is equal to $h\sqrt{1 + 4k^2x^2}\cdot dx$ and this expression is to be used instead of dm in equation (5) of the previous article so that:

$$X = \frac{h\int_{x=a}^{x=b} x\sqrt{1 + 4k^2x^2}\cdot dx}{M} \tag{1}$$

Now the mass of the rod in Fig. 64 is the length of the rod multiplied by h, and the length is given by the integral of $\sqrt{1 + 4k^2x^2}\cdot dx$ from $x = a$ to $x = b$. That is

$$M = h\int_{x=a}^{x=b} \sqrt{1 + 4k^2x^2}\cdot dx \tag{2}$$

When M is determined by this equation its value can be used in equation (1) to give the value X.

In a similar manner equation (6) of Art. 74 gives, for the bent rod shown in Fig. 64:

$$Y = \frac{hk\int_{x=a}^{x=b} x^2\sqrt{1 + 4k^2x^2}\cdot dx}{M} \tag{3}$$

from which Y can be calculated when M is known [see equation (2)].

76. Center of gravity of a solid cone. Consider the solid cone shown in Fig. 66. To determine the location of the center of gravity of this cone equations (5), (6) and (7) of Art. 74 can be used as in the case of a bent rod, but it is worth while to give a slightly different argument as follows: Let T be the torque action of the pull of gravity on the cone, T being reckoned about the axis O (perpendicular to the plane of the paper). Then T is evidently a function of the length x of the cone (x is shown in Fig. 67), and the increment of T due to an increment of x is:

$$dT = x\cdot dF \tag{1}$$

where dF is the force with which gravity pulls on the thin slab of material in Fig. 66, the radius of the slab being y and its thickness dx. The volume of this slab is $\pi y^2 \cdot dx$ and its mass is

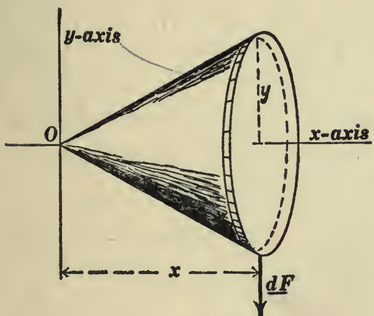


Fig. 66.

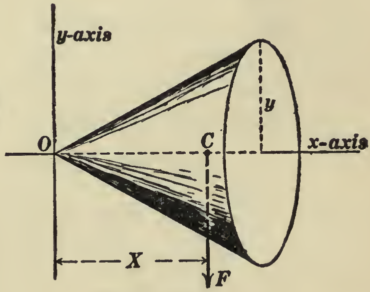


Fig. 67.

$\pi Dy^2 \cdot dx$ $(= dm)$ where D is the density of the material in pounds per cubic foot. But $y = kx$, where k is the tangent of the half-angle of the cone. Therefore $dm = dF = \pi k^2 Dx^2 \cdot dx$, so that equation (1) becomes

$$dT = \pi k^2 Dx^3 \cdot dx \tag{2}$$

whence

$$T = \frac{\pi k^2 D}{4} \cdot x^4 + \text{a constant} \tag{3}$$

but it is evident that T is zero when $x = 0$ because when $x = 0$ there is no cone at all. Therefore the constant of integration in equation (3) is equal to zero, and equation (3) becomes

$$T = \frac{\pi k^2 D}{4} \cdot x^4 \tag{4}$$

Now the volume of the cone in Fig. 66 is a function of x whose increment dV is:

$$dV = \pi y^2 \cdot dx \tag{5}$$

or, using $y = kx$ as before, we have

$$dV = \pi k^2 x^2 dx \tag{6}$$

so that

$$V = \tfrac{1}{3}\pi k^2 x^3 \tag{7}$$

the constant of integration being zero because $V = 0$ when $x = 0$. The mass, M, of the cone is equal to DV pounds, and the total pull of gravity on the cone is therefore $F = DV$ pounds. The torque action of this force about the axis O is

$$FX = DVX = DX \text{ times } \tfrac{1}{3}\pi k^2 x^3$$

and this torque action is equal to T. Therefore, substituting DX times $\frac{1}{3}\pi k^2 x^3$ for T in equation (4) and solving for X, we have:

$$X = \tfrac{3}{4}x \tag{8}$$

77. Average distance of the particles of a body from a plane. A new view of center of gravity. Imagine a body to be made up of minute particles of equal mass. Then the number of particles in a piece of the body would be proportional to the mass of the piece, that is, equal to $N \times$ *mass of piece*, where N is a constant. Let dm be the mass of a small piece of a body; then $N \cdot dm$ is the number of particles in dm. If x is the abscissa of dm, then $x \times N \cdot dm$ is the sum of the abscissas of all the particles in dm, and *the integral* $\int Nx \cdot dm$ *or* $N\int x \cdot dm$ (extended so as to include the entire body under discussion) *is the sum of the abscissas of all the particles of the body.* Furthermore NM is the total number of particles in the body, M being the mass of the body. Therefore, dividing $N\int x \cdot dm$ by NM we have the sum of the abscissas of all the particles divided by the number of particles, which is the *average abscissa of the entire body.* Representing this average abscissa by X we have:

$$X = \frac{N\int x \cdot dm}{NM} = \frac{\int x \cdot dm}{M}$$

Therefore, comparing this with equation (5) of Art. 74 we see that *the abscissa of the center of gravity of a body is the average abscissa of all the particles of the body*, and similar statements can be made with reference to Y and Z as given by equations (6) and (7) of Art. 74. From the point of view of averages the *center of gravity* of a body is usually called the *center of mass* of the body.

PROBLEMS.

1. Four downward forces of 100 pounds, 125 pounds, 200 pounds and 50 pounds act upon a bar at distances of 10 inches, 16 inches, 18 inches and 24 inches from the end of the bar, respectively. Find the value of the resultant of the four forces and the distance from the end of the bar to the point of application of the resultant. Ans. 475 pounds, 16.4 inches.

2. Find the center of gravity of a straight bar 15 inches long and of uniform sectional area on the assumption that the density of the material is given by the equation $D = kx^3$; where D is the density of the material at a point, x is the distance of the point from the end A of the rod, and k is a constant. Ans. 12 inches from end A.

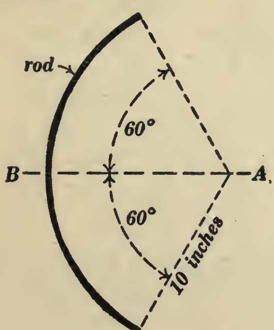


Fig. *p3a*.

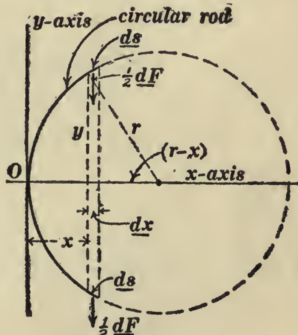


Fig. *p3b*.

3. Find the location of the center of gravity of a rod bent into the arc of a circle as shown in Fig. *p3a*. Ans. 8.27 inches from the center of the circle in the line AB.

Note. Figure $p3b$ shows how this problem may be formulated. The length of the portion ds of the rod is $\frac{dx}{\cos\theta}$ where θ is the angle between ds and the x-axis and it is equal to the complement of the angle between r and the x-axis. That is $\cos\theta = \frac{y}{r}$. The total added mass corresponding to dx is the mass of the two short portions ds. Therefore

$$dm = dF = \frac{2 \cdot dx}{\cos\theta} \times h$$

where h is the mass of the rod per unit length. With this start it is easy to carry the problem through to a conclusion.

4. Find the distance from the 16 inch base to the center of gravity of the entire wedge (30 inches long) which is shown in Fig. $p4$. Ans. 10 inches.

5. Find the distance from the 16 inch base to the center of gravity of the frustum of a wedge (12 inches long) which is shown in Fig. $p4$. Ans. 5.5 inches.

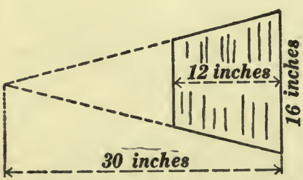


Fig. $p4$.

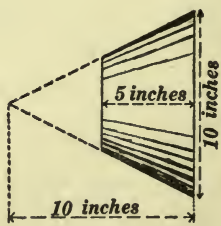


Fig. $p6$.

6. Find the distance from the apex to the center of gravity of the entire cone (10 inches long) shown in Fig. $p6$. Ans. 7.5 inches.

7. Find the distance from the apex to the center of gravity of the frustum of a cone (5 inches long) which is shown in Fig. $p6$. Ans. 8.04 inches.

8. Find the position of the center of gravity of a solid hemi-

sphere whose radius is 16 inches. Ans. 6 inches from the center of the sphere on the axis of symmetry.

Note. This problem can be formulated in a manner very similar to the formulation of the problem of the cone which is discussed in Art. 76; and Fig. *p*8 will suggest the method of formulation.

9. Figure *p*9 represents a segment of a solid sphere. Locate its center of gravity. Ans. 14.32 inches from the center of the sphere on the axis of symmetry of the segment.

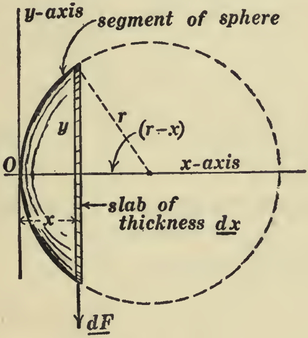


Fig. *p*8.

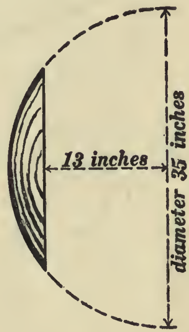


Fig. *p*9.

10. Locate the center of gravity of a thin hemispherical shell whose radius is 16 inches. Ans. 8 inches from the center of the sphere on the axis of symmetry of the shell.

Note. Let Fig. *p*3*b* represent a thin spherical shell. Then $2\pi y . ds$ is the area of the portion of the shell corresponding to dx. The mass dm of this portion of the shell is $2\pi y . ds \times a$, where a is the mass of the shell in pounds per square foot. With this start it is easy to carry the problem through to a conclusion.

11. Locate the center of gravity of the solid paraboloid which is shown in Fig. *p*11. Ans. 8 inches from the vertex of the paraboloid on the axis of the paraboloid.

12. Locate the center of gravity of the frustum of a solid

paraboloid which is shown in Fig. $p12$. Ans. 9.91 inches from the vertex of the paraboloid on the axis.

13. Locate the center of gravity of the thin paraboloidal *shell*

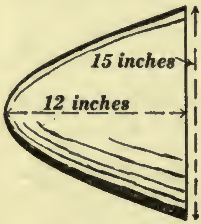


Fig. $p11$.

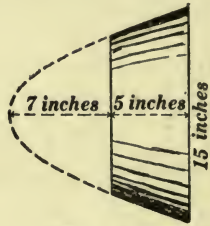


Fig. $p12$.

which is shown in Fig. $p11$. Ans. 6.73 inches from the vertex of the paraboloid on the axis.

Note. In the solution of this problem it is necessary to integrate an expression of the form $\int x\sqrt{a+bx}\cdot dx$. This expression can be put into a form whose integral is easily recognized by substituting

$$z^2 = a + bx$$

so that

$$x = \frac{z^2 - a}{b}$$

and

$$dx = \frac{2zdz}{b}$$

14. Figure $p14$ represents a flat metal plate cut with a parabolic edge. Locate the center of gravity of the whole plate. Ans. 5.4 inches from the vertex of the parabola on the axis.

15. Locate the center of gravity of the portion AB of the plate shown in Fig. $p14$. Ans. $X = 6.66$ inches, $Y = 0$.

16. Locate the center of gravity of the portion A of the plate shown in Fig. $p14$. Ans. $X = 6.66$ inches, $Y = 3.42$ inches.

Note. Consider a small element of the plate as represented by the shaded square in Fig. $p16$. The area of this element is $dx.dy$ and its mass is $dm = k\cdot dx\cdot dy$, where k is a constant. Therefore the integrals $\int x\cdot dm$ and

$\int y \cdot dm$ become $k \int\int x \cdot dx \cdot dy$ and $k \int\int y \cdot dxdy$, respectively. The double integral signs indicate that two integrations are necessary in each case, one with respect to x and the other with respect to y. The limits of these integrations are most easily assigned when the integration with respect to y is made

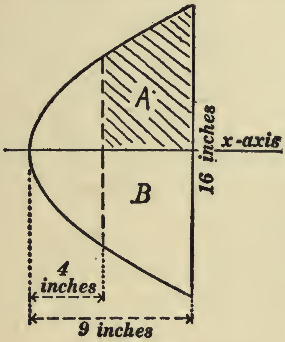


Fig. p14.

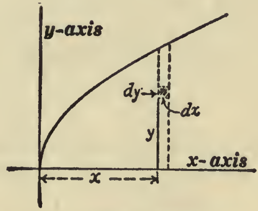


Fig. p16.

first. In this case the integration with respect to y is between the limits $y = 0$ and $y = \sqrt{ax}$ (where $y^2 = ax$ is the equation of the parabola). In this integration x and dx are treated as constants. Then the integration with respect to x is made between the limits $x = 4$ inches and $x = 9$ inches.

17. Locate the center of gravity of a flat plate which is in the shape of a sector of a circle as shown in Fig. p17. Ans. In the line AB at a distance 6.36 inches from A.

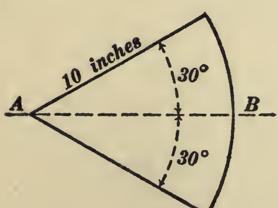


Fig. p17.

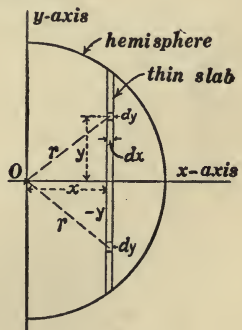


Fig. p18.

18. Suppose that the density of the material of which a sphere is made is kr where k is a constant and r is any distance from the center of the sphere. Locate the center of gravity of half of the sphere, its radius being 2 feet. Ans. $X = 0.8$ foot.

Note. Consider a ring-shaped element of material lying in the thin slab in Fig. $p18$; the radius of the ring being y and the thickness and breadth of the ring being dx and dy respectively. The volume of this ring is $2\pi y \cdot dx \cdot dy$ and its mass is equal to $kr \times 2\pi y \cdot dx \cdot dy$, where $r = \sqrt{x^2 + y^2}$ as shown in the figure. Therefore the pull of gravity (parallel to the y-axis, say) on the ring-shaped element is $2\pi ky\sqrt{x^2 + y^2} \cdot dx \cdot dy$ and the torque action of this pul about the axis O (perpendicular to the plane of the paper) is:

$$dT = 2\pi kxy\sqrt{x^2 + y^2} \cdot dx \cdot dy$$

With this start the problem can be formulated without difficulty. The total mass of the hemisphere can be found by integrating, between proper limits, the expression:

$$dM = 2\pi ky\sqrt{x^2 + y^2} \cdot dx \cdot dy$$

19. What is the average distance of the points on a semi-circle from the diameter which bounds the semi-circle? Ans. 0.64 of the radius.

Note. Let N be the number of points per unit length of a line. Then $N \cdot ds$ is the number of points in the line element ds, and $N\pi r$ is the number of points on the semi-circle.

20. What is the average distance of all the points in a semi-circular area from the diameter which bounds the semi-circle? Ans. 0.42 of the radius.

Note. Let N represent the number of points in one unit of area. Then $N \cdot dA$ is the number of points in an element of area dA and $N \cdot \frac{\pi r^2}{2}$ is the number of points in the semi-circle.

CENTER OF PRESSURE.

78. Force exerted on a water gate. Figure 68 shows a water gate covering an opening in a dam. The short arrows *fff* represent the forces with which the water pushes against the gate. These forces are together equivalent to the single force F, their

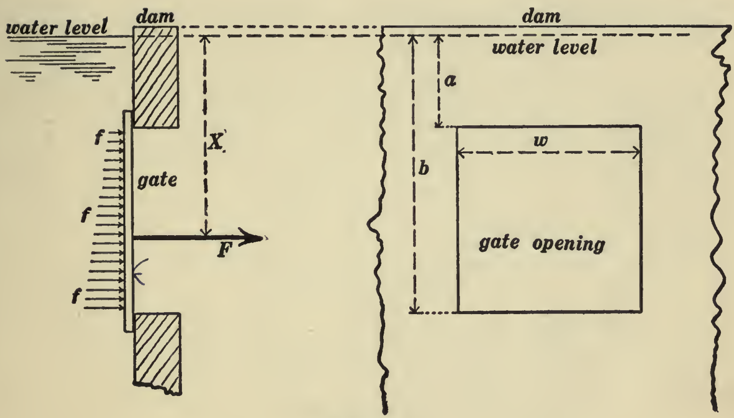


Fig. 68.

resultant, the point of application of which is called the *center of pressure* on the gate. The force F is equal to the sum of all the forces fff, and the torque action of F about an arbitrarily chosen axis O (see Fig. 69) is equal to the sum of the torque actions about O of the forces fff.

Consider a horizontal strip of the gate of which the width is dx as shown in Fig. 69, and of which the length is w. The area of this strip is $w \cdot dx$ and the force dF exerted on the strip is:*

$$dF = Dwx \cdot dx \tag{1}$$

Therefore

$$F = Dw \int_{x=a}^{x=b} x \cdot dx \tag{2}$$

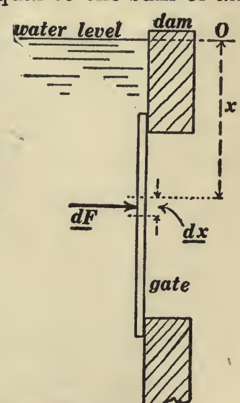


Fig. 69.

* The pressure in pounds per square foot at a place x feet beneath the surface of water is $p = Dx$, where D is the density of water in pounds per cubic foot $(= 62\frac{1}{2})$. Furthermore, the force exerted on the strip is found by multiplying the area of the strip in square feet by the pressure in pounds per square foot.

which gives

$$F = \tfrac{1}{2}Dw(b^2 - a^2) \tag{3}$$

The torque action of the force dF about the axis O is $x \cdot dF$ or $Dwx^2 \cdot dx$, and the total torque action about O of all the forces *fff* is

$$T = Dw\int_{x=a}^{x=b} x^2 \cdot dx \tag{4}$$

which gives

$$T = \tfrac{1}{3}Dw(b^3 - a^3) \tag{5}$$

This torque action must be equal to XF. Therefore using the value of F from equation (3) we have

$$\tfrac{1}{2}Dw(b^2 - a^2)X = \tfrac{1}{3}Dw(b^3 - a^3) \tag{6}$$

whence

$$X = \frac{2}{3}\left(\frac{b^3 - a^3}{b^2 - a^2}\right) \tag{7}$$

79. Force exerted on a curved surface by a stationary fluid under pressure. The short arrows *fff* in Fig. 70 represent the forces with which the water pushes on the curved surface of a dam. All the forces *fff* are together equivalent to a single force F (see Fig. 73) which is called their resultant. Let F_x and F_y be the x and y components of F as shown in Fig. 73. Then F_x is the sum of the x-components of the forces *fff* in Fig. 70, and F_y is the sum of the y-components of the forces *fff*. Let X and Y be the coördinates of the point of application of the resultant force F (see Fig. 73). Then we have the following four conditions which determine F_x, F_y, X and Y:

(*a*) F_x is the sum of the x-components of the forces *fff*.

(*b*) F_y is the sum of the y-components of the forces *fff*.

(*c*) The torque action of F_x about the arbitrarily chosen axis O is F_xY, as may be seen from Fig. 73, and this torque action is equal to the sum of the torque actions about O of the x-components of the forces *fff*.

(*d*) The torque action of F_y about O is F_yX, and this torque action is equal to sum of the torque actions about O of the y-components of the forces *fff*.

In order to formulate these four conditions it is necessary to derive expressions for the x and y components of the force dF which is exerted by a fluid on an element of a curved surface. Consider the surface element AB, Fig. 71, the width of the element being ds and its length l (perpendicular to the plane of the paper). The area of AB is $l.ds$ square feet, and the force in pounds exerted on AB is $dF = pl.ds$, where p is the pressure

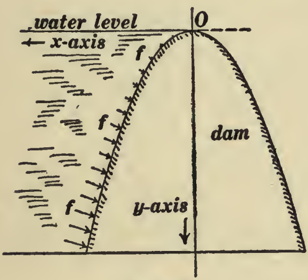


Fig. 70.

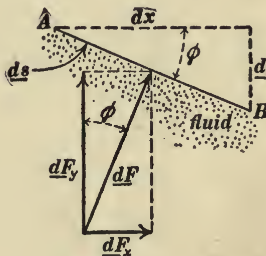


Fig. 71.

of the fluid in pounds per square foot. The x and y components of dF are, respectively:

$$dF_x = \sin \phi \cdot dF$$

and

$$dF_y = \cos \phi \cdot dF$$

Therefore, using $pl \cdot ds$ for dF, using $\frac{dx}{ds}$ for $\cos \phi$, and using $\frac{dy}{ds}$ for $\sin \phi$, we have

$$dF_x = pl \cdot dy \tag{1}$$

and

$$dF_y = pl \cdot dx \tag{2}$$

To determine the total force exerted on the face of the dam in

Fig. 70, that is, to determine the two components F_x and F_y of the total force and the coördinates X and Y of its point of application as shown in Fig. 73, consider the element of area $l \cdot ds$ as shown in Fig. 72, where l is the length of the dam per-

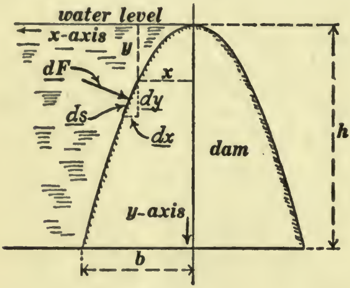


Fig. 72.

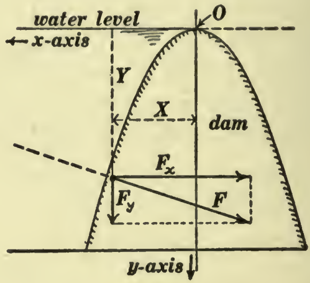


Fig. 73.

pendicular to the plane of the paper. The pressure of the water at this element is Dy pounds per square foot as explained in the footnote to Art. 78. Therefore, using Dy for p in equations (1) and (2), we have for the x and y components of the force dF in Fig. 72:

$$dF_x = Dly \cdot dy \tag{3}$$

and

$$dF_y = Dly \cdot dx \tag{4}$$

From equation (3) we find the x-component of the total force exerted on the dam by integrating between the limits $y = 0$ to $y = h$, which gives:

$$F_x = \tfrac{1}{2} Dlh^2 \tag{5}$$

The equation of the parabola in Fig. 72 is

$$y = \frac{h}{b^2} \cdot x^2$$

Therefore, using this value of y in equation (4) and integrating

between the limits $x = 0$ to $x = b$, we find the y-component of the total force exerted on the dam, namely:

$$F_y = \tfrac{1}{3}Dlhb \tag{6}$$

To find the distance X in Fig. 73 it is necessary to find the total torque action about O of all the forces dF_y and place the result equal to F_yX. The torque action of dF_y (the y-component of dF in Fig. 73) about O is $x.dF_y$, which by equation (4) becomes $Dlxy \cdot dx$. But

$$y = \frac{h}{b^2} \cdot x^2$$

so that the torque action of dF_y about O is $Dl\frac{h}{b^2}x^3 \cdot dx$, and the torque action of all the forces dF_y about O is

$$Dl\frac{h}{b^2}\int_{x=0}^{x=b} x^3 \cdot dx$$

which is equal to $\frac{1}{4}Dlhb^2$. Therefore we have

$$XF_y = \tfrac{1}{4}Dlhb^2 \tag{7}$$

or, using the value of F_y from (6), we have:

$$X = \tfrac{3}{4}b \tag{8}$$

The torque action of dF_x (the x-component of dF in Fig. 73) about O is $y \cdot dF_x$, which by equation (3) becomes $Dly^2 \cdot dy$; and the total torque action of all the forces dF_x is

$$Dl\int_{y=0}^{y=h} y^2 \cdot dy$$

which is equal to $\frac{1}{3}Dlh^3$. Therefore we have:

$$YF_x = \tfrac{1}{3}Dlh^3 \tag{9}$$

or, using the value of F_x from (5), we have

$$Y = \tfrac{2}{3}h \tag{10}$$

PROBLEMS.

1. Find the total force F acting on the gate in Fig. 68 and find the distance X, when $a = 4$ feet, $b = 8$ feet and $w = 4$ feet. Ans. 6000 pounds, 6.22 feet.

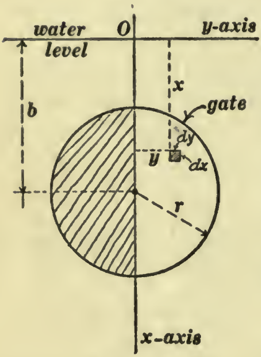


Fig. $p2$.

2. A dam has a circular hole through it 6 feet in diameter. The center of the hole is 7 feet beneath the surface of the water. The hole is covered with a gate. Find the total force with which the water pushes on the gate and find the distance from the surface of the water to point of application of this force. Ans. 12,370 pounds, 7.32 feet.

Note. Consider the element of area which is represented by the small shaded square in Fig. $p2$. The pressure at this area is kx where k is a constant. Therefore the force acting on the element of area is $kx \cdot dx \cdot dy$, and the total force on the gate is $\int\int kx \cdot dx \cdot dy$. To specify the limits of this integration let $y = f(x)$ be the equation of the circle in Fig. $p2$ (the gate). Then if we integrate first with respect to y the limits are $y = -f(x)$ to $y = +f(x)$. We may then integrate with respect to x between the limits $x = b - r$ to $x = b + r$.

The torque action of the force $kx \cdot dx \cdot dy$ about the axis O (perpendicular to the plane of the paper) is $kx^2 \cdot dx \cdot dy$, and the total torque action of the force acting on the entire gate is given by the integral $\int\int kx^2 \cdot dx \cdot dy$. The limits are the same as stated above.

3. Find the abscissa X and the ordinate Y of the center of pressure of the shaded half of the circular water gate in Fig. $p2$, where $b = 7$ feet and $r = 6$ feet. Ans. $X = 7.32$ feet, $Y = 1.27$ feet.

MOMENT OF INERTIA.

80. Kinetic energy of a rotating body. Definition of moment of inertia. Consider a wheel which is rotating n revolutions

per second. Its speed in radians per second is

$$\omega = 2\pi n \tag{1}$$

because there are 2π radians in one revolution. The speed of a rotating body in radians per second is called the *angular velocity* or the *spin velocity* of the body.

Consider a particle of the wheel at a distance r from the axis of rotation. As the wheel rotates this particle travels in a circle of which the circumference is $2\pi r$ and it travels n times round this circle per second so that the velocity v of the particle is

$$v = 2\pi nr \tag{2}$$

or using ω for $2\pi n$ according to equation (1), we have:

$$v = \omega r \tag{3}$$

If the spin velocity of the wheel is doubled it is evident from this equation that the velocity of every particle in the wheel will be doubled so that the kinetic energy of every particle in the wheel will be quadrupled. Therefore the total kinetic energy, W, of a rotating wheel is quadrupled if the spin velocity ω of the wheel is doubled; that is, *for a given wheel,* W is proportional to ω^2, and for the given wheel there is a definite constant by which ω^2 may be multiplied to give W. That is:

$$W = \tfrac{1}{2}K\omega^2 \tag{4}$$

where $(\frac{1}{2}K)$ is the proportionality factor for the given wheel. The constant K is called the *moment of inertia* of the wheel,* and it depends upon the size, shape, and mass of the wheel.

* The kinetic energy of a particle is equal to $\frac{1}{2}mv^2$, where m is the mass of the particle in pounds, v is its velocity in feet per second, and kinetic energy is expressed *not* in foot-pounds, but in *foot-poundals*. Throughout this discussion of moment of inertia distance is expressed in feet, velocity in feet per second, mass in pounds, force in poundals, torque in poundal-feet, work or energy in foot-poundals, and moment of inertia in pound-feet.[2]

81. General integral expression for moment of inertia. Imagine a small particle of mass dm to be added to the spinning body at a distance r from the axis of spin. Then the velocity v of the added particle is ωr, and the kinetic energy of the added particle is $\frac{1}{2}dm \times \omega^2 r^2$ which is, of course, the infinitesimal increment of the kinetic energy of the spinning body due to the added particle. Therefore

$$dW = \tfrac{1}{2}\omega^2 r^2 dm$$

and by integration* we have

$$W = \tfrac{1}{2}\omega^2 \int r^2 \cdot dm \tag{5}$$

Comparing this equation with equation (4) it is evident that

$$K = \int r^2 \cdot dm \tag{6}$$

82. Average value of the square of the distances of all the particles of a body from an axis. Definition of radius of gyration. Using the ideas of Art. 77, $N \cdot dm$ is the number of particles in a small piece dm of a body. If r is the distance of the small piece dm from a chosen axis, then $Nr^2 \cdot dm$ is the sum of the squares of the distances of all the particles in dm from the axis, and the integral $\int Nr^2 . dm$ or $N\int r^2 \cdot dm$ extended so as to include an entire body is the sum of the squares of the distances of all the particles of the body from the axis. But NM is the number of particles in the body. Therefore

$$\frac{N\int r^2 \cdot dm}{NM} = \frac{\int r^2 \cdot dm}{M} \tag{1}$$

is the average value of the squares of the distances of all the particles of a body from the axis. This average may be repre-

* This integration can only be indicated. Before the actual value of the integral can be found r and dm must be expressed in terms of one or more independent variables, and the limits of the integration must be such as to include the entire spinning body.

sented by ρ^2 so that

$$\rho^2 = \frac{\int r^2 \cdot dm}{M} \tag{2}$$

or

$$\rho^2 M = \int r^2 \cdot dm \tag{3}$$

but $\int r^2 \cdot dm$ is the moment of inertia K of the body with respect to the chosen axis, according to equation (6) of Art. 81. Therefore we have:

$$K = \rho^2 M \tag{4}$$

The distance ρ, which is the square-root-of-the-average-value-of-the-squares-of-the-distances-of-all-the-particles-of-a-body-from-a-chosen-axis, is called the *radius of gyration* of the body with respect to the chosen axis.

83. Moment of inertia of a circular saw about its axis of spin. The moment of inertia of a circular saw could be easily derived from equation (6) of Art. 81, but it is instructive to give the complete argument again as follows: Let W be the kinetic energy of a circular disk r feet in radius rotating at a constant spin velocity of ω radians per second, as shown in Fig. 74, and

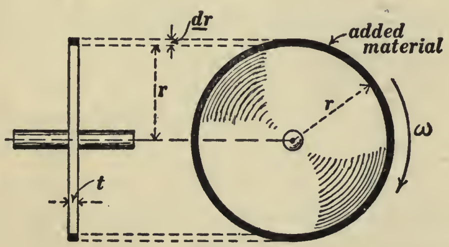


Fig. 74.

let it be required to find the infinitesimal increment of W due to an arbitrary infinitesimal increment of r. Let t be the thickness of the disk and let D pounds per cubic foot be the density of the material of which the disk is made. Imagine the radius

of the spinning disk to be increased by the addition of material as indicated in Fig. 74. The volume of the added material is $2\pi r \times t \times dr$, the mass of the added material is $2\pi tr \cdot dr \times D$ pounds, the velocity of the added material is ωr, the kinetic energy of the added material is equal to one half the product of its mass times the square of its velocity, and the kinetic energy of the added material is the desired infinitesimal increment dW. Therefore:

$$dW = \pi D t \omega^2 r^3 \cdot dr \tag{1}$$

W and r being the only variables. Therefore, by integration we get:

$$W = \tfrac{1}{4}\pi D t \omega^2 r^4 + \text{a constant} \tag{2}$$

But W must evidently be zero when r is zero. Therefore the constant of integration must be equal to zero, so that equation (2) becomes

$$W = \tfrac{1}{4}\pi D t \omega^2 r^4 \tag{3}$$

Now $\pi r^2 t D$ is equal to the mass m of the disk in pounds so that equation (3) becomes:

$$W = \tfrac{1}{4}\omega^2 m r^2 \tag{4}$$

But according to Art. 80 the kinetic energy of any rotating body can be expressed as $\frac{1}{2}K\omega^2$, where K is the moment of inertia of the body. Therefore we have

$$W = \tfrac{1}{4}\omega^2 m r^2 = \tfrac{1}{2}K\omega^2 \tag{5}$$

from which we have:

$$K = \tfrac{1}{2}mr^2 \tag{6}$$

That is, the moment of inertia of a circular saw about its axis of spin is equal to one half the mass of the saw in pounds multiplied by the square of the radius of the saw.

Remark. The thickness t in the above discussion may be anything whatever. Therefore equation (6) expresses the moment of inertia of a circular cylinder of any length rotating about its axis of figure.

84. Moment of inertia of a rectangular bar. To determine the moment of inertia of the rectangular bar shown in Fig. 75 the general equation (6) of Art. 81 will be used. Figure 76 represents a top view of the bar, O being the axis of rotation. Consider the element of the bar which is represented by the small black square in Fig. 76. This element is understood to extend entirely through the bar parallel to the axis O (at right angles to the plane of the paper in Fig. 76). Therefore the volume of the element is $t \cdot dx \cdot dy$ and its mass is $tD \cdot dx \cdot dy$. The distance of the element from the axis is $r = \sqrt{x^2 + y^2}$, so that

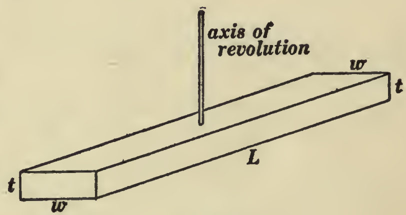


Fig. 75.

$$r^2 = x^2 + y^2$$

Therefore the general equation (6) of Art. 81 becomes:

$$K = tD\int\int(x^2 + y^2) \cdot dx \cdot dy \tag{1}$$

in which the double integral sign is used because two integrations, one with respect to x and one with respect to y are necessary, as explained in the note to problem 16 on page 132.

If we integrate with respect to y (treating x and dx as constants) between the limits $y = -\frac{w}{2}$ to $y = +\frac{w}{2}$ we get the moment of inertia with respect to the axis O of the thin slab between the dotted lines in Fig. 76. We may then integrate with respect to x between the limits $x = -\frac{l}{2}$ to $x = +\frac{l}{2}$

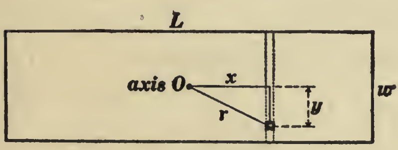


Fig. 76.

and we get the desired expression for K, namely:

$$K = \frac{Dtwl}{12}(l^2 + w^2) \tag{2}$$

But $Dtwl$ is the mass m of the bar in pounds, so that equation (2) becomes:

$$K = \frac{1}{12}(l^2 + w^2)m \tag{3}$$

Remark. The same final result is obtained if we integrate first with respect to x (treating y and dy as constants), and then with respect to y. What is the meaning of the result of the first integration in this case?

85. Torque required to increase the spin-velocity of a body. The velocity of a particle in a rotating wheel is $v = r\omega$, according to equation (3) of Art. 80. Therefore when ω increases, v increases r times as fast as ω. That is:

$$\frac{dv}{dt} = r\frac{d\omega}{dt} \tag{1}$$

This $\frac{dv}{dt}$ is the acceleration of the particle in the direction at right angles to r*, and

$$\frac{dv}{dt}\cdot dm \left(= r\frac{d\omega}{dt}\cdot dm\right)$$

is the sidewise force (at right angles to r) which must act on the particle to produce the sidewise acceleration. The torque action of this sidewise force about the axis of rotation is:

$$dT = r\frac{d\omega}{dt}\cdot dm \times r$$

or

$$dT = \frac{d\omega}{dt}\cdot r^2\cdot dm \tag{2}$$

* We are not here concerned with the radial acceleration of the particle which is discussed in Art. 50.

so that the total torque action required to increase the spin velocity of the body at the rate $\frac{d\omega}{dt}$ is:

$$T = \frac{d\omega}{dt}\int r^2 \cdot dm \tag{3}$$

The integral is of course understood to be extended over the whole body and it is equal to the moment of inertia of the body according to equation (6) of Art. 81. Therefore equation (3) may be written

$$T = K\frac{d\omega}{dt} \tag{4}$$

That is, the spin acceleration $\frac{d\omega}{dt}$ of a body multiplied by the moment of inertia of the body is equal to the torque which must be exerted on the body to produce the spin acceleration.

86. Moments of inertia about parallel axes. Let K be the moment of inertia of a body about an axis (perpendicular to the plane of the paper in Fig. 77) which passes through the center of gravity O of the body, and let K' be the moment of inertia of the body about an axis P (perpendicular to the plane of the paper) which is at a distance a from O. Then

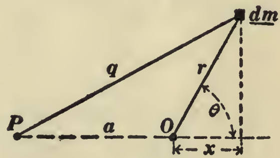


Fig. 77.

$$K' = K + a^2M \tag{1}$$

where M is the total mass of the body in grams or pounds as the case may be.

Proof. From the triangle in Fig. 77 we have:

$$q^2 = a^2 + r^2 + 2ar\cos\theta$$

or

$$q^2 = a^2 + r^2 + 2ax \tag{2}$$

where x is the abscissa of the element of material dm referred

to the center of gravity O as an origin. Now according to equation (6) of Art. 81 we have

$$K' = \int q^2 \cdot dm \tag{3}$$

Therefore, using the value of q^2 from equation (2), we have:

$$K' = \int a^2 \cdot dm + \int r^2 \cdot dm + \int 2ax \cdot dm$$

or

$$K' = a^2 \int dm + \int r^2 \cdot dm + 2a \int x \cdot dm \tag{4}$$

But $\int dm$ = the total mass M of the body, $\int r^2 \cdot dm$ is the moment of inertia K referred to the axis O, and $\int x \cdot dm$ is zero according to equation (5) of Art. 74, because the center of gravity O in Fig. 77 is taken as the origin. Therefore equation (4) becomes:

$$K' = a^2M + K \tag{1}$$

87. Moment of section of a beam. Consider equation (2) of Art. 72, namely:

$$dT = \frac{2Eb}{R} \cdot x^2 \cdot dx \tag{1}$$

The product $2b \cdot dx$ is the area of the shaded strips in Fig. 59, and x is the distance of this area from the axis OO. Let us use dA for $2b \cdot dx$, then equation (1) becomes:

$$dT = \frac{E}{R} x^2 \cdot dA \tag{2}$$

so that

$$T = \frac{E}{R} \int x^2 \cdot dA \tag{3}$$

The integral $\int x^2 \cdot dA$ is called the *moment of section* of the beam; and according to equation (3) the torque T which bends any beam is equal to $\frac{ES}{R}$, where E is the stretch modulus of the material of the beam, R is the radius of curvature of the median line of the beam, and $S\left(=\int x^2 \cdot dA\right)$ is the *moment of section* of the beam.

The integral $\int x^2 \cdot dA$ is similar in form to the integral $\int r^2 \cdot dm$ in equation (6) of Art. 81, and because of this similarity engineers call $\int x^2 \cdot dA$ the "*moment of inertia*" *of the section of the beam.* The term *moment of section* is however the correct term.

PROBLEMS.

1. Find the moment of inertia of a circular saw six feet in diameter and of which the mass is 125 pounds. Ans. 562.5 lb.-ft.^{2}

2. Find the amount of energy in foot-pounds stored in the saw of problem 1 at a speed of 600 revolutions per minute. Ans. 34482 foot-pounds.

Note. In the formula $W = \frac{1}{2}K\omega^2$, K is expressed in pound-feet-squared, ω is expressed in radians per second, and W is expressed in foot-poundals (not in foot-pounds). There are 32.2 foot-poundals in one foot-pound.

3. Find the moment of inertia of a cylinder 2 feet in diameter and 3 feet long with respect to its axis, the density of the material being 420 pounds per cubic foot. Ans. 1980 lb.-ft.^{2}

4. Find the moment of inertia of a hollow-cylinder, the external dimensions being the same as in problem 3, inside diameter being 1 foot, the axis of revolution being the axis of the cylinder, and the density of the material being 420 pounds per cubic foot. Ans. 1114 lb.-ft.^{2}

5. Find the moment of inertia of the rectangular bar in Figs. 75 and 76. The bar is 5 feet long, 1 foot wide and 0.5 foot thick and it has a mass of 1000 pounds. Ans. 2167 lb.-ft.^{2}

6. Find the moment of inertia of the bar in problem 5 when the axis of rotation coincides with the edge t in Fig. 75. Ans. 8667 lb.-ft.^{2}

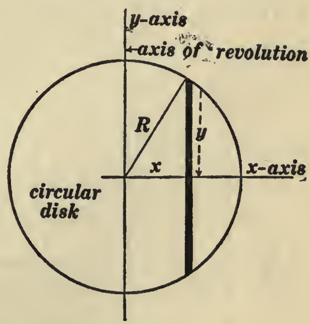


Fig. p7.

7. Find the moment of inertia of a very thin circular disk when

the axis of rotation is a diameter of the disk; the mass of the disk being 120 pounds and its diameter being 6 feet. Ans. 270 lb.-ft.^{2}

Note. Let the circle in Fig. *p*7 represent the disk. Take the narrow vertical black strip as dm. Then $dm = k \times 2y \cdot dx$ where k is pounds per square foot of disk area.

8. Find the moment of inertia of the thin circular disk with respect to the axis O as shown in Fig. *p*8, the density of the

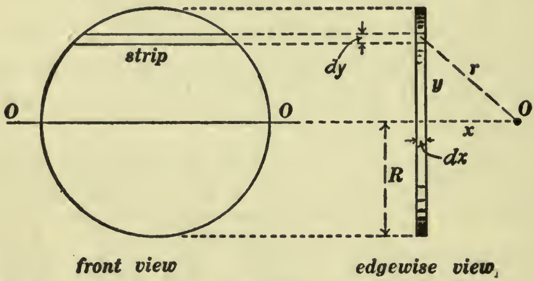


Fig. *p*8.

material being D. Ans. $K = \pi R^2 D\left(\frac{R^2}{4} + x^2\right) \cdot dx.$

Note. The mass of the narrow strip in Fig. *p*8 is

$$2D\sqrt{R^2 - y^2} \cdot dx \cdot dy = dm$$

and the distance of the strip from the axis is

$$\sqrt{x^2 + y^2} = r$$

Of course x and dx are constants in this problem.

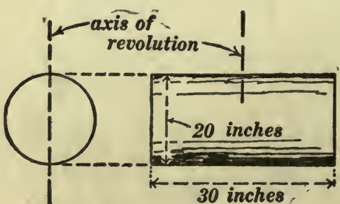


Fig. *p*9.

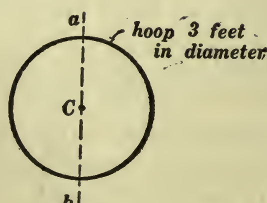


Fig. *p*10.

9. Find the moment of inertia of the solid cylinder shown in Fig. p9; the density of the material being 0.28 pound per cubic inch. Ans. 1833 lb.-ft.^{2}

Note. The method of formulating this problem is suggested by problem 8.

10. A slender circular ring or hoop has a mass of 10 pounds and it is 3 feet in diameter. (a) Find its moment of inertia about the axis C (perpendicular to the plane of the paper in Fig. p10), and (b) find its moment of inertia about the axis ab. Ans. (a) 22.5 lb.-ft.^{2} (b) 11.25 lb.-ft.^{2}

11. Find the moment of inertia of a solid steel sphere 10 inches in diameter, the axis of revolution being a diameter of the sphere. The density of steel is 0.28 pound per cubic inch. Ans. 22.4 lb.-ft.^{2}

Note. Let the circle in Fig. p11 represent the sphere. Take for dm a thin cylindrical shell of radius r and thickness dr.

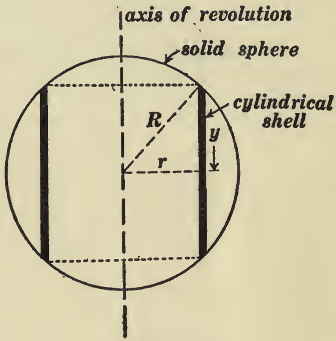


Fig. p11.

12. Find the moment of inertia of a steel governor ball, 4 inches in diameter, with respect to the axis of revolution of the governor, the center of the ball being 6 inches from the axis. Ans. 2.45 lb.-ft.^{2}

TABLE OF MOMENTS OF INERTIA.

Axis through center of mass in each case.
m = mass of body in grams or pounds.
K = moment of inertia.

1. *Thin straight bar* of length l and having uniform section, axis at right angles to bar. $K = \frac{1}{12}l^2m$.

2. *Rectangular parallelopiped,* axis parallel to one edge, a and b being lengths of other edges. $K = \frac{1}{12}(a^2 + b^2)m$

3. *Cylinder or disk* of radius r, referred to axis of cylinder. $K = \frac{1}{2}r^2m$.

Referred to axis at right angles to axis of cylinder

$$K = \left(\frac{l^2}{12} + \frac{r^2}{4}\right) m$$

where l is the length of the cylinder.

4. *Hollow cylinder* of radii R and r, referred to axis of cylinder. $K = \frac{1}{2}(R^2 + r^2)m$.

5. *Sphere* of radius r referred to a diameter. $K = \frac{2}{5}r^2m$.

Note. If the axis of rotation does not pass through the center of mass, use equation (1) of Art. 86.

CHAPTER VI.

EXPANSIONS IN SERIES. USE OF COMPLEX QUANTITY.

88. Maclaurin's theorem.—Let y be any function whatever of x which is finite and continuous and of which all of the derivatives with respect to x are finite and continuous. Then:

$$\mathbf{y} = \mathbf{A} + \mathbf{Bx} + \frac{1}{2}\mathbf{Cx}^2 + \frac{1}{\lfloor 3}\mathbf{Dx}^3 + \frac{1}{\lfloor 4}\mathbf{Ex}^4 + \cdots \qquad (1)$$

where A, B, C, D, etc., are constants as follows:

A is the value of y when $x = 0$

B is the value of $\frac{dy}{dx}$ when $x = 0$

C is the value of $\frac{d^2y}{dx^2}$ when $x = 0$

D is the value of $\frac{d^3y}{dx^3}$ when $x = 0$

E is the value of $\frac{d^4y}{dx^4}$ when $x = 0$

etc., etc.

Equation (1) expresses what is known as *Maclaurin's theorem.*

Proof.—Let the curve cc in Fig. 78 represent the given function. Then the value of y which is to be expressed by equation (1) is the ordinate of the point p. To establish equation (1) we will make a series of approximations and consider the limit towards which this series of approximations trends, as follows:

First approximation.—To get a first approximation let us assume that $\frac{dy}{dx}$ is equal to the constant B (the value of $\frac{dy}{dx}$

when $x = 0$) everywhere between the points p and q in Fig. 78. That is by assumption we have:

$$\frac{dy}{dx} = B \tag{2}$$

Integrating this differential equation we have:

$$y = Bx + \text{a constant}$$

but $y = A$ when $x = 0$, so that the constant of integration is

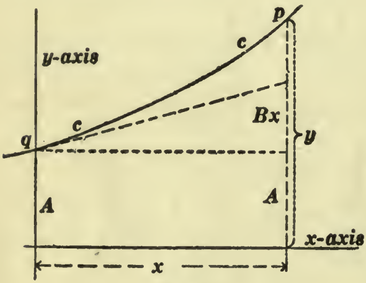


Fig. 78.

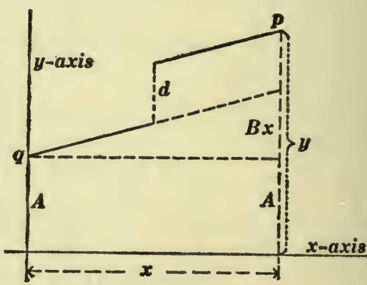


Fig. 79.

evidently equal to A. Therefore as a first approximation we have:

$$\mathbf{y = A + Bx} \tag{3}^*$$

It is interesting to note that to assume $\frac{dy}{dx} = B$ everywhere is the same thing as to take the inclined dotted line in Fig. 78 as an approximation to the curve cc; and the ordinate of this inclined straight line is $A + Bx$.

* Figure 79 shows a curve pq for which $\frac{dy}{dx} = B$ everywhere, but the ordinate of p is not equal to $A + Bx$ because of the discontinuity or jump at d. The function y and all of its derivatives must be finite and continuous everywhere between p and q, as stated at the beginning. Any discontinuity outside of the region between p and q does not vitiate equation (1) for the region between p and q.

Second approximation.—To get a second approximation let us assume that $\frac{d^2y}{dx^2}$ is equal to the constant C (the value of $\frac{d^2y}{dx^2}$ when $x = 0$) everywhere between p and q in Fig. 78. That is, by assumption, we have:

$$\frac{d^2y}{dx^2} = C \tag{4}$$

Integrating once we have:

$$\frac{dy}{dx} = Cx + \text{a constant}$$

But $\frac{dy}{dx} = B$ when $x = 0$, so that the constant of integration is equal to B, giving:

$$\frac{dy}{dx} = B + Cx \tag{5}$$

Integrating again we have:

$$y = Bx + \frac{1}{2}Cx^2 + \text{a constant}$$

But $y = A$ when $x = 0$, so that the constant of integration is equal to A, giving as a second approximation:

$$\mathbf{y = A + Bx + \frac{1}{2}Cx^2} \tag{6}$$

Third approximation.—To get a third approximation let us assume that $\frac{d^3y}{dx^3}$ is equal to the constant D (the value of $\frac{d^3y}{dx^3}$ when $x = 0$) everywhere between p and q in Fig. 78. That is, by assumption, we have:

$$\frac{d^3y}{dx^3} = D \tag{7}$$

By three successive integrations (the constant of each integra-

tion being determined as above) we get, as a third approximation:

$$\mathbf{y} = \mathbf{A} + \mathbf{Bx} + \frac{1}{2}\,\mathbf{Cx}^2 + \frac{1}{\underline{|3}}\,\mathbf{Dx}^3 \tag{8}$$

nth approximation.—To get an nth approximation let us assume that $\frac{d^ny}{dx^n}$ is equal to the constant N (the value of $\frac{d^ny}{dx^n}$ when $x = 0$) everywhere between p and q in Fig. 78. That is, by assumption, we have:

$$\frac{d^ny}{dx^n} = N \tag{9}$$

By n successive integrations this differential equation gives, as the nth approximation:

$$\mathbf{y} = \mathbf{A} + \mathbf{Bx} + \frac{1}{2}\mathbf{Cx}^2 + \frac{1}{\underline{|3}}\,\mathbf{Dx}^3 + \ldots + \frac{1}{\underline{|n}}\,\mathbf{Nx}^n \tag{10}$$

To show that the nth approximation approaches the true value of y as a limit as n approaches infinity. In the first place it is evident that equation (10) gives equation (1) when n is increased more and more, but it remains to be shown that y in equation (10) approaches the correct value of y as a limit as n is increased.

The nth derivative $\frac{d^ny}{dx^n}$ is assumed to be everywhere finite and it must have therefore a definite *largest value* V and a definite *smallest value* v between p and q. If the largest value V is used instead of N in equation (10) we get too large a value, a, for y. If the smallest value v is used instead of N in equation (10) we get too small a value, b, for y.* That is:

$$a = A + Bx + \frac{1}{2}\,Cx^2 + \cdots \frac{1}{\underline{|n}} \cdot Vx^n \tag{11}$$

* This statement happens to be plausible and therefore the reader is apt to accept it as true without actually perceiving its truth, which is indeed evident if one takes the trouble to think about it.

and

$$b = A + Bx + \frac{1}{2} Cx^2 + \cdots \frac{1}{\underline{|n}} \cdot vx^n \tag{12}$$

and the true value of y lies between a and b. But, subtracting equation (12) from equation (11) member by member we get:

$$a - b = \frac{(V - v)x^n}{\underline{|n}} \tag{13}$$

and this difference approaches zero as n approaches infinity, because when n is increased by 1 the numerator is multiplied by the finite quantity x, whereas the denominator is multiplied by the quantity $n + 1$, which becomes as large as you please.

It is evident therefore that the value of a as given by equation (11) becomes more and more nearly equal to the value of b as given by equation (12). But the true value of y lies between a and b. Therefore equations (11) and (12) both approach the true value of y as a limit as n is increased; and equations (11) and (12) reduce to equation (1) when n is indefinitely great.

Taylor's series.—Heretofore any algebraic expression containing x has been spoken of as a function of x. Thus $cx + e$ is a function of x, c and e being costants. Also such expressions as $e^{(x+h)}$, $\sin(x + h)$, $\tan(x + h)$ are functions of x. If, however, x is a constant and h a variable we would think of these expressions as functions of h.

Let y be any function whatever of $(y + h)$, let x be a constant and h a variable, and let it be understood that y and all of its derivatives are finite and continuous. Then y may be expanded by Maclaurin's theorem, giving:

$$y\,[\text{any function of } (x + h)] = A + Bh + \frac{1}{\underline{|2}} Ch^2 + \frac{1}{\underline{|3}} Dh^3 + \cdots \tag{14}$$

where A, B, C, D, etc., are constants, as follows:

A is the value of y when $h = 0$

$$B \text{ is the value of } \frac{dy}{dh} \text{ when } h = 0$$

$$C \text{ is the value of } \frac{d^2y}{dh^2} \text{ when } h = 0$$

$$\text{etc.,} \qquad\qquad \text{etc.}$$

Equation (14) is sometimes called *Taylor's series*, but it is identical to *Maclaurin's series* and to give it a separate name is to create a false distinction.

89. Examples. (*a*) *Expansion of* e^x.—The successive derivatives of e^x are as follows:

$$y = e^x \text{ which is equal to 1 when } x = 0$$

$$\frac{dy}{dx} = e^x \text{ which is equal to 1 when } x = 0$$

$$\frac{d^2y}{dx^2} = e^x \text{ which is equal to 1 when } x = 0$$

$$\text{etc.,} \qquad\qquad \text{etc.}$$

Therefore $A = B = C = D = \text{etc.} = 1$, and equation (1) of Art. 88 gives:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{\underline{|3}}x^3 + \frac{1}{\underline{|4}}x^4 + \text{etc.} \tag{1}$$

(*b*) *Expansion of* $\sin x$.—The successive derivatives of $\sin x$ are as follows:

$$y = \sin x \text{ which is equal to } 0 \text{ when } x = 0$$

$$\frac{dy}{dx} = \cos x \text{ which is equal to } 1 \text{ when } x = 0$$

$$\frac{d^2y}{dx^2} = -\sin x \text{ which is equal to } 0 \text{ when } x = 0$$

$$\frac{d^3y}{dx^3} = -\cos x \text{ which is equal to } -1 \text{ when } x = 0$$

$$\frac{d^4y}{dx^4} = \sin x \text{ which is equal to } 0 \text{ when } x = 0$$

and so on in endless repetition.

Therefore $A = 0$, $B = 1$, $C = 0$, $D = -1$, $E = 0$, etc., and equation (1) of Art. 88 gives:

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} \text{ etc.} \tag{2}$$

(*c*) *Expansion of* $\cos x$.—The successive derivatives of $\cos x$ are as follows:

$$\begin{aligned}
y &= \cos x \text{ which is equal to } 1 \text{ when } x = 0\\
\frac{dy}{dx} &= -\sin x \text{ which is equal to } 0 \text{ when } x = 0\\
\frac{d^2y}{dx^2} &= -\cos x \text{ which is equal to } -1 \text{ when } x = 0\\
\frac{d^3y}{dx^3} &= \sin x \text{ which is equal to } 0 \text{ when } x = 0\\
\frac{d^4y}{dx^4} &= \cos x \text{ which is equal to } 1 \text{ when } x = 0
\end{aligned}$$

and so on in endless repetition.

Therefore $A = 1$, $B = 0$, $C = -1$, $D = 0$, $E = 1$, etc., and equation (1) of Art. 88 gives:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} \text{ etc.} \tag{3}$$

90. Maclaurin's theorem applied to a function of two independent variables.—If u is a function of x and y which is itself finite and continuous, and if all of its derivatives (partial derivatives) are finite and continuous, then:

$$\begin{aligned}
u = {} & A\\
& + B_x x + B_y y\\
& + \frac{1}{2}(C_{xx}x^2 + 2C_{xy}xy + C_{yy}y^2)\\
& + \frac{1}{\underline{|3}}(D_{xxx}x^3 + 3D_{xxy}x^2y + 3D_{xyy}xy^2 + D_{yyy}y^3) \qquad (1)\\
& \qquad \text{etc.} \qquad \text{etc.} \qquad \text{etc.}
\end{aligned}$$

Where A is the value of u when x and y are both zero.

B_x is the value of $\frac{du}{dx}$ when x and y are both zero.

B_y is the value of $\frac{du}{dy}$ when x and y are both zero.

C_{xx} is the value of $\frac{d^2u}{dx^2}$ when x and y are both zero.

C_{xy} is the value of $\frac{d^2u}{dxdy}$ or $\frac{d^2u}{dydx}$ when x and y are both zero.

C_{yy} is the value of $\frac{d^2u}{dy^2}$ when x and y are both zero.

etc. etc. etc.

The proof of Maclaurin's theorem as applied to a function of two or more variables is essentially identical to the proof of the theorem as applied to a function of one variable. To enable one to appreciate the modified character of argument and especially as an interesting example of partial integration, let us consider what we may call the second approximation, that is the expression we get for u on the assumption that the respective second derivatives are everywhere constant and equal to C_{xx}, C_{xy} and C_{yy} (their respective values when $x = 0$ and $y = 0$) as follows:

$$\frac{d^2u}{dx^2} = C_{xx} \tag{2}$$

$$\frac{d^2u}{dxdy} = C_{xy} \tag{3}$$

$$\frac{d^2u}{dy^2} = C_{yy} \tag{4}$$

Integrating equation (2) with respect to x and equation (3) with respect to y we have

$$\frac{du}{dx} = C_{xx}x + \text{any function of } y \tag{5}$$

and

$$\frac{du}{dx} = C_{xy}y + \text{any function of } x \tag{6}$$

Now $\frac{du}{dx}$ is equal to B_x when x and y are both equal to zero. Therefore the *constant term* in "any function of y" in (5) is equal to B_x, and the constant term in "any function of x" in (6) is equal to B_x. Therefore, using the

expression "vanishing function of y" for a function of y which is equal to zero when $y = 0$, equations (5) and (6) become:

$$\frac{du}{dx} = C_{xx}x + (\text{a vanishing function of } y) + B_x \tag{7}$$

and

$$\frac{du}{dx} = C_{xy}y + (\text{a vanishing function of } x) + B_x \tag{8}$$

But these two expressions must be identical, consequently the "vanishing function of y" in (7) must be $C_{xy}y$ and the "vanishing function of x" in (8) must be $C_{xx}x$. Therefore both of these equations reduce to

$$\frac{du}{dx} = C_{xx}x + C_{xy}y + B_x \tag{9}$$

In a similar manner we may integrate equation (4) with respect to y and equation (3) with respect to x, and get the equation:

$$\frac{du}{dy} = C_{yy}y + C_{xy}x + B_y \tag{10}$$

Now equations (9) and (10) may be integrated, giving:

$$u = \tfrac{1}{2}C_{xx}x^2 + C_{xy}xy + B_xx + \text{any function of } y \tag{11}$$

and

$$u = \tfrac{1}{2}C_{yy}y^2 + C_{xy}xy + B_yy + \text{any function of } x \tag{12}$$

But $u = A$ when x and y are both equal to zero, therefore "any function of y" in (11) must be ("a vanishing function of y") $+ A$, and likewise, "any function of x" in (12) must be ("a vanishing function of x") $+ A$. Furthermore, equations (11) and (12) must be identical so that the "vanishing function of y" in (11) must be $\frac{1}{2}C_{yy}y^2 + B_yy$, and the "vanishing function of x" in (12) must be $\frac{1}{2}C_{xx}x^2 + B_xx$. Therefore equations (11) and (12) both reduce to:

$$\begin{aligned} u = A & \\ & + B_xx + B_yy \\ & + \tfrac{1}{2}(C_{xx}x^2 + 2C_{xy}xy + C_{yy}y^2) \end{aligned} \tag{13}$$

which is the approximate value of u as derived from equations (2), (3) and (4).

PROBLEMS.

1. Expand $\log(1 + x)$ by Maclaurin's theorem. Ans.

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Note.—The function $\log x$ and all of its derivatives become infinite for $x = 0$, and therefore $\log x$ cannot be expanded in powers of x.

2. Using the answer to problem 1 make an attempt to calculate the logarithm of 0 by placing $x = -1$ and adding together a number of terms of the series.

Note.—The series obtained by Maclaurin's theorem for $\log(1 + x)$ cannot be used for values of x which lead up to or beyond a point where $\log(1 + x)$ or any of its derivatives become discontinuous or infinite.

3. Expand $\cos(x + h)$ in a series of ascending powers of h. Ans.

$$\cos(x + h) = \cos x - h \sin x - \frac{h^2}{\underline{|2}} \cos x + \frac{h^3}{\underline{|3}} \sin x + \cdots$$

4. Expand $y = \tan x$ by Maclaurin's theorem. Ans.

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Note.—When $x = \frac{\pi}{2}$ the given function and its derivatives become infinite. Therefore the series found in answer to this problem does not give the value of $\tan x$ for values of x equal to or greater than $\frac{\pi}{2}$.

91. Demoivre's Theorem.—An important algebraic identity, which was discovered by Demoivre, is expressed by the equation:

$$e^{jx} = \cos x + j \sin x \tag{1}$$

where e is the Napierian base and $j = \sqrt{-1}$.* This relation is known as *Demoivre's theorem*, and it is very useful in certain transformations for purposes of integration, and it is also useful in the theory of alternating currents.

To establish equation (1) write jx for x in equation (1) of Art. 89, remembering that $j^2 = -1$, $j^3 = -j$, $j^4 = +1$, etc., and we have:

$$e^{jx} = 1 - \frac{x^2}{2} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \text{etc.} + j\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{etc.}\right) \tag{2}$$

* It is customary in the theory of alternating currents to use i for electric current, and j is used for $\sqrt{-1}$.

But the real terms in this series give a series identical to equation (3) of Art. 89, and the imaginary terms give a series identical to equation (2) of Art. 89. Therefore from equation (2) we get $e^{jx} = \cos x + j \sin x$.

Definition of complex quantity. Geometric representation of complex quantity.—Any expression like $a + b\sqrt{-1}$, which is part real and part imaginary is called a *complex quantity*. Thus the right-hand member of equation (1) is a complex quantity; and of course e^{jx} is a complex quantity because equation (1) shows that e^{jx} is part real $(\cos x)$ and part imaginary $(j \sin x)$.

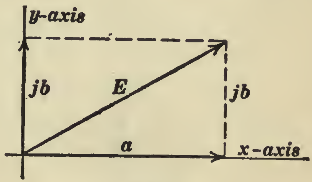


Fig. 80.

The accepted method of representing a complex quantity geometrically is shown in Fig. 80. The vector E is the complex quantity, its x-component is thought of as a real quantity a, its y-component is thought of as the imaginary quantity jb, and the vector is the sum of its two components. That is

$$E = a + jb^*$$

* This algebraic expression of a vector as a complex quantity is one aspect of the important use of complex quantity in the theory of alternating currents. See Franklin and Esty's *Elements of Electrical Engineering*, Vol. II, Chapter V; The Macmillan Co., New York, 1908.

Another aspect of the use of complex quantity in the theory of alternating currents is exhibited in Chapter VII of this treatise where the fundamental differential equations of alternating currents are integrated with the help of transformations involving the use of complex quantity.

The practical use of complex quantity in alternating-current theory is due chiefly to C. P. Steinmetz; but the use of complex quantity in the solution of linear differential equations as explained in Chapter VII contains everything that is now known of the use of complex quantity in alternating-current theory in a manner which is self-evident, and the use of complex quantity as exemplified in Chapter VII is much older than electrical engineering.

92. Euler's expressions for $\sin x$ and $\cos x$.—Many differential expressions can be reduced to simple recognizable forms for purposes of integration with the help of Demoivre's theorem, and it is in some cases convenient to modify equation (1) of Art. 91 so as to express $\sin x$ and $\cos x$ in terms of exponentials Such expressions are due to Euler and they are:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{1}$$

and

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{2}$$

The use of these equations is exemplified below. They are derived as follows: From Demoivre's theorem we have:

$$e^{jx} = \cos x + j \sin x \tag{3}$$

Write $-x$ for x in this expression, remembering that

$$\cos(-x) = \cos x$$

and that

$$\sin(-x) = -\sin x$$

and we have:

$$e^{-jx} = \cos x - j \sin x \tag{4}$$

Subtracting equation (3) from equation (4) member from member we get equation (1), and adding equations (3) and (4) member to member we get equation (2).

93. Example showing use of Euler's equations.—Consider the function:

$$z = \sin mx \cos nx \tag{1}$$

To find the average value of this function between $x = 0$ and $x = 2\pi$ it is necessary to integrate $z \cdot dx$ between the limits $x = 0$ and $x = 2\pi$ as explained in Art. 73. Now the function given in equation (1) is a function whose average value is of fundamental importance in connection with Fourier's theorem (see Chapter VIII) and therefore it is important to be able to

reduce the differential expression $\sin mx \cos nx \cdot dx$ to a combination of fundamental forms which can be found in Class A of the table of integrals in appendix B. This reduction may be easily made with the help of Euler's equations as follows:

It is required to integrate the differential equation:

$$dy = \sin mx \cos nx \cdot dx \tag{2}$$

Writing mx for x in Euler's expression for $\sin x$ we get an expression for $\sin mx$, and writing nx for x in Euler's expression for $\cos x$ we get an expression for $\cos nx$. Substituting these expressions for $\sin mx$ and $\cos nx$ in equation (2), we get:

$$dy = \frac{1}{4j} e^{j(m+n)x} \cdot dx + \frac{1}{4j} e^{j(m-n)x} \cdot dx \\ - \frac{1}{4j} e^{-j(m+n)x} \cdot dx - \frac{1}{4j} e^{-j(m-n)x} \cdot dx \tag{3}$$

Each term in the second member of this equation is of the form $ae^{bx} \cdot dx$ of which the integral (ignoring constant of integration) is $\frac{a}{b} \cdot e^{bx}$. Thus, in case of the first term $a = \frac{1}{4j}$ and

$$b = j(m + n)$$

so that the integral of the first term is

$$-\frac{1}{4(m+n)} e^{j(m+n)x}$$

Proceeding in a similar manner with each term and arranging the result systematically we get:

$$y = \frac{1}{4(m+n)} [e^{-j(m+n)x} - e^{j(m+n)x}] \\ + \frac{1}{4(m-n)} [e^{-j(m-n)x} - e^{j(m-n)x}] \tag{4}$$

And this expression can be easily reduced to form 24 in the table of integrals by using Euler's equations.

PROBLEMS.

1. Reduce $dy = \sin^3 x \cdot dx$ to a combination of standard forms as given in the table of integrals. Ans.

$$dy = \frac{1}{8j}(e^{-j3x} - e^{j3x} + 3e^{jx} - 3e^{-jx}) \cdot dx$$

2. Reduce $dy = \cos^4 x \cdot dx$ to a combination of standard forms. Ans.

$$dy = \tfrac{1}{16}(e^{j4x} + e^{-j4x} + 4e^{j2x} + 4e^{-j2x} + 6) \cdot dx$$

94. Hyperbolic sines and cosines.*—In the solution of the differential equation of the alternating-current transmission line† the expressions $(e^x - e^{-x})$ and $(e^x + e^{-x})$ occur, and transmission line calculations are facilitated by the use of tables giving the values of $(e^x - e^{-x})$ and $(e^x + e^{-x})$ for various values of x, and of course the discussion of such calculations is simplified by having names for $(e^x - e^{-x})$ and $(e^x + e^{-x})$. Indeed the expressions $\dfrac{e^x - e^{-x}}{2}$ and $\dfrac{e^x + e^{-x}}{2}$ are related to the equilateral hyperbola in the same way that $\dfrac{e^{jx} - e^{-jx}}{2j}$ $(= \sin x)$ and $\dfrac{e^{jx} + e^{-jx}}{2}$ $(= \cos x)$ are related to the circle. Therefore these expressions are called the *hyperbolic sine of* x ($\sinh x$) and the

* Tables giving values of a, b, c and d in the expressions

$$\cosh(x + jy) = a + jb$$

and

$$\sinh(x + jy) = c + jd$$

for various values of x and y are published in a supplement to the *General Electric Review* for May, 1910.

† The simplest discussion of this subject is that which is given in Franklin's *Electric Waves*, pages 141–153, The Macmillan Co., New York, 1909. This discussion is essentially complete, although no mention is made of hyperbolic sines and cosines.

hyperbolic cosine of x (cosh x) respectively. That is:

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{1}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{2}$$

PROBLEMS.

1. Differentiate $y = \sinh x$.
2. Differentiate $y = \cosh x$.
3. Integrate $dy = \sinh x \cdot dx$.
4. Integrate $dy = \cosh x \cdot dx$.

For answers see forms 29 and 30 of table of integrals in Appendix B.

CHAPTER VII.

SOME ORDINARY DIFFERENTIAL EQUATIONS.

95. Degree and order of a differential equation.—The simple equation $az + b = 0$ is said to be *linear* with respect to z because it contains no power of z higher than the first power. A differential equation of the form:

$$y + A\frac{dy}{dx} + B\frac{d^2y}{dx^2} + C = 0 \tag{1}$$

is called a *first degree* or *linear* differential equation because it contains no products of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and no powers of y, $\frac{dy}{dx}$, etc., higher than the first. The coefficients A, B and C in a linear differential equation may contain the independent variable x, but we shall confine our attention in this chapter chiefly to *linear differential equations with constant coefficients.*

If the first derivative,* only, appears in a differential equation, the differential equation is said to be of the *first order.* If the second derivative occurs (with or without the first derivative) the differential equation is said to be of the *second order;* and so on.

96. Ordinary and partial differential equations.—A differential equation expressing the law of growth of a function of one independent variable is called an *ordinary differential equation.* Many examples of ordinary differential equations are given in Chapters I and V. See Art. 24 in particular.

* The terms *degree* and *order* apply to partial differential equations as well as to ordinary differential equations. A function of two variables, however, has two first derivatives, three second derivatives, and so on as explained in Art. 59. Therefore it is somewhat misleading to speak of *the* first derivative, or *the* second derivative in explaining what is meant by a differential equation of the first or second order.

A differential equation which expresses the law of growth of a function of two or more independent variables is called a *partial differential equation.* Some examples of partial differential equations are given in Chapter V. See Arts. 60 and 90 in particular.

This chapter is devoted to the discussion of a few important ordinary differential equations, and several important partial differential equations are discussed in Chapters VIII and IX.

97. Pure and mixed differential equations.*—A *pure differential equation* contains but one derivative (the first or any higher derivative) and does not contain the dependent variable y. Thus $dy = 6x \cdot dx$, $\frac{dy}{dx} = \sin x$, $\frac{d^2y}{dx^2} = \log x$ are pure differential equations.

A mixed differential equation contains more than one derivative, or it contains one or more derivatives and the dependent variable y. Thus

$$\frac{dy}{dx} = y, \quad y + B\frac{dy}{dx} = Cx^2, \quad B\frac{dy}{dx} + C\frac{d^2y}{dx^2} = 0$$

are mixed differential equations.

Solution of pure differential equations.—A pure differential equation can be integrated by looking up the appropriate form in the table of integrals. This is exemplified by every problem in integration heretofore given in this treatise. In the case of a pure differential equation of the second or third order, successive integrations are of course necessary. For example, consider the third order pure differential equation:

$$\frac{d^3y}{dx^3} = ax^2 \tag{1}$$

Let q represent $\frac{d^2y}{dx^2}$, then equation (1) becomes:

$$\frac{dq}{dx} = ax^2 \tag{2}$$

* This and the following articles refer primarily to ordinary differential equations.

This equation can be integrated, giving:

$$q = \frac{d^2y}{dx^2} = \tfrac{1}{3}ax^3 + C \tag{3}$$

Let p represent $\frac{dy}{dx}$, then equation (3) becomes:

$$\frac{dp}{dx} = \tfrac{1}{3}ax^3 + C \tag{4}$$

This equation can be integrated, giving:

$$p = \frac{dy}{dx} = \tfrac{1}{12}ax^4 + Cx + D \tag{5}$$

This equation can be integrated, giving

$$y = \tfrac{1}{48}ax^5 + \tfrac{1}{2}Cx^2 + Dx + E \tag{6}$$

Solution of mixed differential equations. Separation of variables.—The linear differential equation (1) of Art. 95 is of course a mixed equation, and the general solution of this equation with constant coefficients is given in a subsequent article. We will here consider one or two examples of a simple transformation, called *the separation of variables,* which can sometimes be used to bring a mixed differential equation into a form which can be integrated by looking up appropriate forms in the table of integrals.

As a first example consider:

$$\frac{dy}{dx} = x^2y \tag{7}$$

This equation reduces to:

$$\frac{dy}{y} = x^2 \cdot dx \tag{8}$$

and this equation can be integrated with the help of forms 1 and 2 of the table of integrals, giving

$$\log y = \tfrac{1}{3}x^3 + C \tag{9}$$

As a second example consider

$$\frac{d^2y}{dx^2} = x^2\frac{dy}{dx} \tag{10}$$

Let p represent $\frac{dy}{dx}$ and this equation becomes

$$\frac{dp}{dx} = x^2p \tag{12}$$

and the integral of this equation, according to equation (9), is:

$$\log p = \tfrac{1}{3}x^3 + C$$

or

$$p = \frac{dy}{dx} = e^{\frac{1}{3}x^3+C} \tag{13}$$

This is a pure differential equation and its integral can be found by looking up the appropriate forms in the table of integrals.

PROBLEMS.

Solve the following differential equations:

1. $x^3\frac{d^3y}{dx^3} = 2.$ Ans. $y = \log x + Cx^2 + Dx + E.$
2. $\frac{d^2y}{dx^2} = xe^x.$ Ans. $y = (x - 2)e^x + Cx + D.$
3. $\frac{d^3y}{dx^3} = 27\sin^3 x.$ Ans. $y = 21\cos x - \cos^3 x + Cx^2 + Dx + E.$
4. $1 - y + (1 + x)\frac{dy}{dx} = 0.$ Ans. $\log\frac{1+x}{1-y} = C.$
5. $1 - 2y = 3\frac{dy}{dx}.$ Ans. $C(1 - 2y) = e^{-\frac{2x}{3}}.$
6. $\sin y\cdot dx + x\cos y\cdot dy = 0.$ Ans. $x\sin y = C.$
7. $\sin x\cos y\cdot dx - \cos x\sin y\cdot dy = 0.$ Ans. $\cos y = C\cos x.$
8. $(y + xy)dx + (x - xy)dy = 0.$ Ans. $\log xy = C - x + y.$
9. $\sqrt{1 - y^2}\cdot dx + \sqrt{1 - x^2}\cdot dy = 0.$ Ans. $\sin^{-1} x + \sin^{-1} y = C.$
10. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 0.$ Ans. $C\log x = y + D.$

11. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 1 = 0.$ Ans. $y = \log \cos (x - C) + D.$

12. $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + ax = 0.$

Ans. $y = D - ax + C \log [x + \sqrt{1 + x^2}].$

98. The general solution and particular solutions of a differential equation.—The general solution of a differential equation is an expression for y (the dependent variable) in terms of x (the independent variable) which includes every possible* function which satisfies the differential equation. Thus equation (6) of Art. 97 is the general solution of equation (1) of that article. Concerning the three constants of integration it is evident that E is the value of y when $x = 0$. Also it is evident from equation (5) of Art. 97 that D is the value of $\dfrac{dy}{dx}$ when $x = 0$, and it is evident from equation (3) that C is the value of $\dfrac{d^2y}{dx^2}$ when $x = 0$.

The *general solution* of a differential equation of the nth order contains n undetermined constants of integration.

When one or more of the constants of integration in the general solution of a differential equation have particular values assigned to them we have what is called a *particular solution* of the differential equation. For example, let $C = 2$, let $D = 0$ and let $E = 0$ in equation (6) of Art. 97, then the equation becomes $y = \dfrac{ax^5}{48} + x^2$ which is a particular solution of equation (1) of Art. 97.

99. Discussion of the first order linear differential equation with constant coefficients.—Consider the differential equation:

$$\mathbf{y} + \mathbf{A}\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{0} \qquad (1)$$

* This statement is subject to some qualification because of what is called the *singular solution*. See Johnson's *Ordinary and Partial Differential Equations*, page 43, John Wiley & Sons, New York, 1890.

The most obvious method for solving this differential equation is to "separate the variables" as explained in Art. 97. Thus equation (1) is easily reduced to:

$$A\frac{dy}{y} = -dx \tag{2}$$

and this equation can be integrated by looking up appropriate forms in the table of integrals, giving:

$$A \log y = -x + C \tag{3}$$

It is desirable, however, to solve equation (1) by the following method because the method applies to a linear differential equation of any order when the coefficients are constants.

Let

$$\mathbf{y} = \mathbf{C}\mathbf{e}^{kx} \tag{4}$$

where C and k are constants, and e is the Napierian base. Then:

$$\frac{dy}{dx} = kCe^{kx} \tag{5}$$

Substituting the values of y and $\frac{dy}{dx}$ from (4) and (5) in equation (1), we have:

$$Ce^{kx} + AkCe^{kx} = 0 \tag{6}$$

whence by cancellation we have:

$$1 + Ak = 0$$

or

$$\mathbf{k} = -\frac{\mathbf{1}}{\mathbf{A}} \tag{7}$$

Therefore if k in equation (4) has the value $-\frac{1}{A}$ then equation (4) satisfies equation (1); that is, equation (4) is a solution of (1) if $k = -\frac{1}{A}$; indeed equation (4) is the general solution of

(1) because it contains one undetermined constant C, the constant of integration.

100. The principle of superposition.—A principle of extremely wide application in physics is the so-called *principle of super-position*. From the physical point of view a general statement of this principle is scarcely possible and therefore the following examples must suffice: (*a*) A person at A (Fig. 81) can see window No. 1 and another person at B can see window No. 2 *at the same time*. This means that two beams of light a and b can travel through the same region at the same time and not get tangled up together as it were, each beam behaving as if it were traveling through the region alone. (*b*) Two sounds can travel through the same body of air simultaneously, each sound behaving as if it were traveling through the body of air alone. (*c*) Two systems of water waves can travel over the same part of a pond simultaneously, each system behaving as if the other were not present. (*d*) Two messages* can travel over a telegraph wire at the same time and not get mixed up. (*e*) Two forces F and G exerted simultaneously upon an elastic structure produce an effect which is the sum of the effects which would be produced by the forces separately, provided the sum of the forces does not exceed the elastic limit of the structure; therefore each force may be thought of as producing the same effect that it would produce if it were acting alone

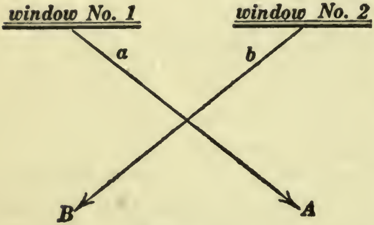


Fig. 81.

All of the effects in physics which are superposable—and this

* Indeed any number of messages can travel over a telegraph wire in either direction or in both directions simultaneously. The only limiting feature in multiplex telegraphy, when line-loss is negligible, is in the design of the sending and receiving apparatus; and the same is true in wireless telegraphy. In each of the above examples the word *two* means *two or more*.

includes the greater part of the effects in mechanics, heat, electricity and magnetism, light and sound, and a great many effects in chemistry—are expressible in terms of linear differential equations with constant coefficients, and the principle of superposition may be thought of as a property of such a differential equation as follows. If y *is a function of* x *which satisfies a linear differential equation with constant coefficients, and if* z *is another function of* x *which satisfies the same differential equation, then* $(y + z)$ *is a function of* x *which satisfies the equation.**

Proof.—Let the given linear differential equation be:

$$u + A\frac{du}{dx} + B\frac{d^2u}{dx^2} + \cdots = 0 \tag{1}$$

If a function y satisfies this equation, then:

$$y + A\frac{dy}{dx} + B\frac{d^2y}{dx^2} + \cdots = 0 \tag{2}$$

If another function z satisfies equation (1), then:

$$z + A\frac{dz}{dx} + B\frac{d^2z}{dx^2} + \cdots = 0 \tag{3}$$

Now $\dfrac{d(y + z)}{dx} = \dfrac{dy}{dx} + \dfrac{dz}{dx}$ and $\dfrac{d^2(y + z)}{dx^2} = \dfrac{d^2y}{dx^2} + \dfrac{d^2z}{dx^2}$. Therefore, adding equations (2) and (3), we get:

$$(y + z) + A\frac{d(y + z)}{dx} + B\frac{d^2(y + z)}{dx^2} + \cdots = 0 \tag{4}$$

But this is the same form as equation (1) which shows that the function $(y + z)$ satisfies (1).

* This proposition is true for a partial linear differential equation also. Indeed most of the superposable effects in physics are expressible in terms of partial linear differential equations with constant coefficients. Examples are given in Chapters VIII and IX.

101. Discussion of the second order linear differential equation with constant coefficients.—Consider the differential equation:

$$y + A\frac{dy}{dx} + B\frac{d^2y}{dx^2} = 0 \tag{1}$$

It is not possible to "separate the variables" in this differential equation and therefore the second method of Art. 99 must be used. Therefore let:

$$y = Ce^{kx} \tag{2}$$

then

$$\frac{dy}{dx} = kCe^{kx} \tag{3}$$

and

$$\frac{d^2y}{dx^2} = k^2Ce^{kx} \tag{4}$$

Substituting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1) and cancelling the common factor Ce^{kx}, we have:

$$1 + Ak + Bk^2 = 0 \tag{5}$$ A E

whence

$$k = -\frac{A}{2B} \pm \sqrt{\frac{A^2}{4B^2} - 1} \tag{6}$$

Therefore using α for one of these values of k and using β for the other value of k, we get two solutions of (1), namely:

$$w = Ce^{\alpha x} \tag{7}$$

and

$$z = De^{\beta x} \tag{8}$$

But according to Art. 100 the sum $(w + z)$ is also a solution. Therefore using y for $(w + z)$, we have as a solution of (1):

$$y = Ce^{\alpha x} + De^{\beta x} \tag{9}$$ generales

and this is the general solution of (1) because it contains the two undetermined constants C and D.

102. The starting of a boat.—At a certain instant ($t = 0$) a constant force E begins to act on a boat, and it is desired to find an expression for the increasing velocity i of the boat. At very low speeds the backward drag or friction of the water on a boat is proportional to the velocity of the boat. *Let us assume that this proportional relation is exact,* then the frictional drag of the water on a boat is equal to Ri, where i is the velocity of the boat and R is a constant which depends on the shape and size of the boat. Therefore the net accelerating force acting on the boat is $E - Ri$. This force is equal* to the product of the mass L of the boat and the acceleration $\frac{di}{dt}$. Therefore we have:

$$L\frac{di}{dt} = E - Ri \tag{1}$$

To get this equation into the standard form of a linear differential equation, let

$$y = E - Ri \tag{2}$$

Then

$$\frac{dy}{dt} = -R\frac{di}{dt}$$

and equation (1) becomes:

$$-\frac{L}{R} \cdot \frac{dy}{dt} = y$$

or

$$y + \frac{L}{R} \cdot \frac{dy}{dt} = 0 \tag{3}$$

The general solution of this equation is given in Art. 99, but it is worth while to work it out anew, as follows: Let

$$y = Ce^{kt} \tag{4}$$

* If force is expressed in poundals (or dynes), mass in pounds (or grams), and acceleration in feet per second per second (or centimeters per second per second).

then

$$\frac{dy}{dt} = kCe^{kt} \tag{5}$$

Substituting these values of y and $\frac{dy}{dt}$ in equation (3) and cancelling out the factor Ce^{kt}, we have:

$$1 + \frac{L}{R} \cdot k = 0$$

so that

$$k = -\frac{R}{L} \tag{6}$$

Therefore equation (4) becomes:

$$y = Ce^{-\frac{R}{L} \cdot t}$$

or, substituting $E - Ri$ for y, we have:

$$E - Ri = Ce^{-\frac{R}{L} \cdot t} \tag{7}$$

Now $i = 0$ when $t = 0$. Therefore placing this pair of values in equation (7) we have:

$$E = C \tag{8}$$

so that the constant of integration is determined, and equation (7) becomes:

$$E - Ri = Ee^{-\frac{R}{L} \cdot t}$$

or

$$i = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L} \cdot t} \tag{9}$$

This equation expresses the value at each instant of the increasing velocity of the boat as a function of the elapsed time t.

103. The stopping of a boat.—A boat is moving at velocity I when the propelling force ceases to act, and it is required to find an expression for the decreasing velocity, i, of the boat as it

gradually comes to rest. In this case the only force acting on the boat is the retarding force Ri, and in this case the acceleration $\frac{di}{dt}$ is negative so that:

$$L\frac{di}{dt} = -Ri \tag{1}$$

or

$$i + \frac{L}{R}\cdot\frac{di}{dt} = 0 \tag{2}$$

The general solution of this differential equation is:

$$i = Ce^{-\frac{R}{L}\cdot t} \tag{3}$$

But $i = I$ when $t = 0$. Therefore $C = I$, and equation (3) becomes:

$$i = Ie^{-\frac{R}{L}\cdot t} \tag{4}$$

Remark.—The notation used in this discussion of the starting and stopping of a boat is the standard notation used in electrical theory. This notation is used here and in the following articles

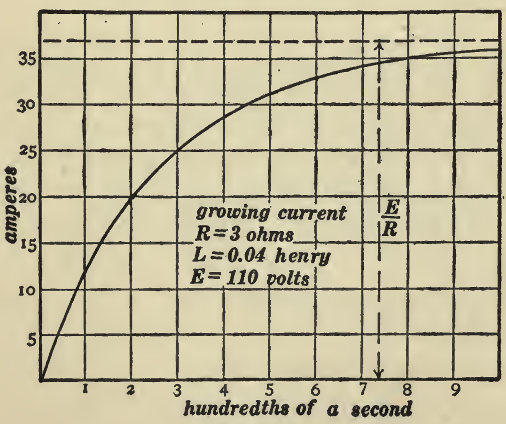


Fig. 82.

because the mechanical problems discussed in this and the following articles are strictly analogous to certain electrical problems which constitute the foundation of the theory of alternating currents. Thus the curve in Fig. 82 is a graphical representation of equation (9) of Art. 102; the ordinates of this curve represent the growing values of the current i in a circuit

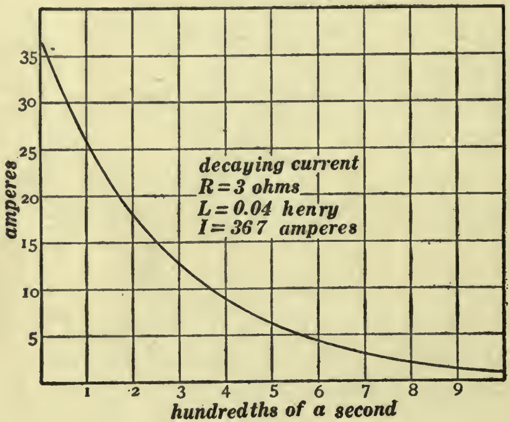


Fig. 83.

of resistance R and inductance L after a battery of electromotive force E is connected to the circuit. The curve in Fig. 83 is a graphical representation of equation (4) of Art. 103; the ordinates of this curve represent the decaying values of the current i when an initial current of I amperes is left to die away in a circuit of resistance R and inductance L.

PROBLEMS.

Solve the following differential equations.

1. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0.$ Ans. $y = Ce^{-x} + De^{2x}.$

2. $\dfrac{d^2y}{dx^2} = 4y.$ Ans. $y = Ce^{2x} + De^{-2x}.$

3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 0.$ Ans. $y = Ce^{x} + De^{3x}.$

4. $3\left(\dfrac{d^2y}{dx^2} + y\right) = 10\dfrac{dy}{dx}.$ Ans. $y = Ce^{3x} + De^{\frac{1}{3}x}.$

5. $\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 0.$ Ans. $y = Ce^{2x} + De^{3x} + E.$

104. Undamped oscillations.—Figure 84 shows a weight suspended by a spring, and the weight stands in equilibrium in a certain position. If the weight happens to be q feet above or below its equilibrium position an unbalanced force proportional to q, and therefore equal to $\frac{1}{C}$* times q acts upon the weight. This force is downwards when q is upwards, and upwards when q is downwards. Therefore the force is equal to $-\frac{1}{C}\cdot q$. This force accelerates (or retards) the weight, and we have

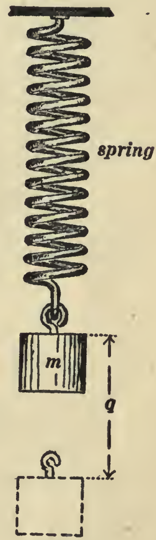


Fig. 84.

$$L\frac{di}{dt} = -\frac{q}{C} \tag{1}$$

where L is the mass in pounds of the attached weight in Fig. 84, and $\frac{di}{dt}$ is the acceleration (it is a retardation when it is negative) of the attached weight. But the velocity i of the weight is the rate of change of its distance q from the equilibrium position. That is

$$i = \frac{dq}{dt} \tag{2}$$

so that

$$\frac{di}{dt} = \frac{d^2q}{dt^2} \tag{3}$$

* This is simply a proportionality factor. It is written in this form so a to conform to the standard notation of electrical theory.

Therefore equation (1) becomes

$$L\frac{d^2q}{dt^2} = -\frac{q}{C}$$

or

$$q + \frac{L}{C}\cdot\frac{d^2q}{dt^2} = 0 \tag{4}$$

The general solution of this second order linear differential equation is given in Art. 101, but it is worth while to work out the solution anew as follows:
Let

$$q = Me^{kt} \tag{5}$$

then

$$\frac{dq}{dt} = kMe^{kt} \tag{6}$$

and

$$\frac{d^2q}{dt^2} = k^2Me^{kt} \tag{7}$$

Substituting these values of q and $\frac{d^2q}{dt^2}$ in equation (4) and cancelling the common factor Me^{kt}, we have:

$$1 + \frac{L}{C}\cdot k^2 = 0 \tag{8}$$

so that

$$k = \sqrt{-\frac{L}{C}} \tag{9}$$

It is evident that k is imaginary because L and C are positive quantities. Therefore let us write $j\omega$ for k where j is $\sqrt{-1}$,* then:

$$\omega = \sqrt{\frac{L}{C}} \tag{9}$$

* It is customary in electrical theory to use j for $\sqrt{-1}$ because the letter i is used for electric current.

Now of course ω may be either positive or negative, that is, k may be $+j\omega$ or $-j\omega$. Therefore

$$v = Me^{+j\omega t} \tag{10}$$

and

$$z = Ne^{-j\omega t} \tag{11}$$

are two particular solutions of equation (4), where M and N are undetermined constants. Therefore, according to Art. 100, $(v + z)$ is also a solution of (4). Consequently, writing u for $(v + z)$, we have as a solution of (4):

$$u = Me^{+j\omega t} + Ne^{-j\omega t} \tag{12}$$

and this is the general solution of (4) because it contains the two undetermined constants M and N.

Now from Demoivre's theorem it is evident that u as given by equation (12) is complex (part real and part imaginary), and for the sake of generality M and N may be thought of as complex also. That is, for M and N we may write:

$$M = M' + jM'' \tag{13}$$

and

$$N = N' + jN'' \tag{14}$$

Now any complex equation is equivalent to two simple equations, namely the equation which expresses the equality of the real parts of the members of the complex equation, and the equation which expresses the equality of the imaginary parts of the complex equation; and this latter equation becomes real when j is cancelled from its members. Thus equation (12) may be broken up into two simple equations each of which is a solution of equation (4).

To split equation (12) up into two equations use the values for M and N from equations (13) and (14), and reduce $e^{+j\omega t}$ and $e^{-j\omega t}$ by means of Demoivre's theorem as explained in Art. 91. We thus get:

$$\begin{aligned} r + js = (M' + jM'')&(\cos \omega t + j \sin \omega t) \\ &+ (N' + jN'')(\cos \omega t - j \sin \omega t) \end{aligned} \tag{15}$$

where $r + js$ has been written for q. Separating real and imaginary terms in equation (15) we get:

$$r = (M' + N') \cos \omega t - (M'' - N'') \sin \omega t \tag{16}$$

and

$$s = (M'' + N'') \cos \omega t + (M' - N') \sin \omega t \tag{17}$$

Now in either of these equations the coefficient of $\cos \omega t$ is *any* constant, and the coefficient of $\sin \omega t$ is *any* constant. Therefore, q for r or s, both of these equations reduce to:

$$q = G \cos \omega t + H \sin \omega t \tag{18}$$

which is the final general solution of equation (4), G and H being undetermined constants.

In alternating current theory this equation is used in a slightly modified form as follows:

Let

$$G = -Q \sin \theta \tag{19}$$

and

$$H = Q \cos \theta \tag{20}$$

Then equation (18) becomes:

$$q = Q \sin (\omega t - \theta) \tag{21}$$

This equation is represented by Fig. 85, in which the line Q rotates at angular velocity ω radians per second, and the value of q at each instant is represented by the projection of Q on the fixed line ab.

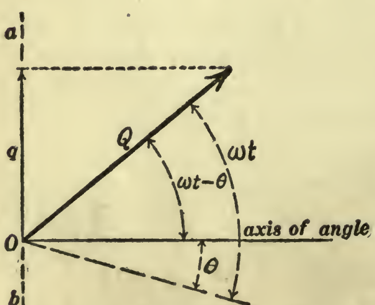


Fig. 85.

The two integration constants Q and θ are determined by the initial conditions as follows: For example let the weight be started oscillating by pulling it upwards 2 feet ($q = 2$ feet),

holding it still $\left(\frac{dq}{dt} = 0\right)$ and releasing it at the instant $t = 0$. The values of Q and θ are then determined by substituting the values $\frac{dq}{dt} = 0$ and $t = 0$ in the equation

$$\frac{dq}{dt} = \omega Q \cos(\omega t - \theta)$$

and by substituting the values $q = 2$ and $t = 0$ in equation (21).

105. Damped oscillations.—The weight in Fig. 84 is acted upon by the force $-\frac{q}{C}$ when it is at a distance q above or below its equilibrium position as explained in Art. 104; and, if the weight moves up and down through a resisting fluid, the backward drag due to friction will be approximately proportional to the velocity $\frac{dq}{dt}$ of the weight. Assuming this proportional relationship to be exact, the backward drag of friction may be written $R \cdot \frac{dq}{dt}$, and since it is always opposed to the velocity $\frac{dq}{dt}$ it must be written $-R\frac{dq}{dt}$. Therefore the total force acting on the weight is $-\frac{q}{C} - R\frac{dq}{dt}$, and this force is equal to the product of the mass L of the weight and the acceleration $\frac{d^2q}{dt^2}$ of the weight. Therefore we have:

$$L\frac{d^2q}{dt^2} = -\frac{q}{C} - R\frac{dq}{dt}$$

or

$$q + CR\frac{dq}{dt} + LC\frac{d^2q}{dt^2} = 0 \tag{1}$$

The general solution of this equation may be found by the method used in Art. 104. Reduced to simple form the general

solution is:

$$q = Qe^{-\alpha t} \sin(\omega t - \theta) \tag{2}$$

where Q and θ are undetermined constants of integration and where α and ω are written for the following:

$$\alpha = \frac{R}{2L} \tag{3}$$

and

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{4}$$

Equation (2) completely expresses the motion of the weight in Fig. 84 when the motion of the weight is opposed by friction as above explained. Indeed equation (2) expresses a kind of harmonic motion in which the amplitude $(Qe^{-\alpha t})$ decreases continually.

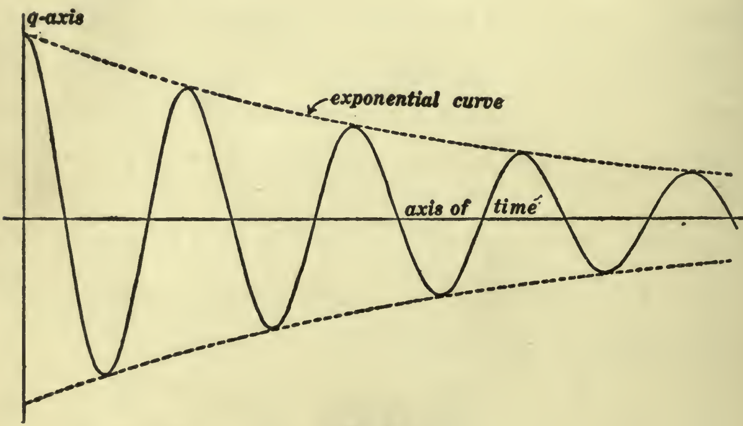


Fig. 86.

The varying value of q as expressed by equation (2) is represented by the ordinates of the wavy curve in Fig. 86 for the case in which $\theta = -\frac{\pi}{2}$. The dotted curves are exponential curves

of which the equations are $q = \pm Qe^{-at}$. Another graphical representation of equation (2) is shown in Fig. 87 where q is the projection on the fixed line ab of a line Qe^{-at} which rotates

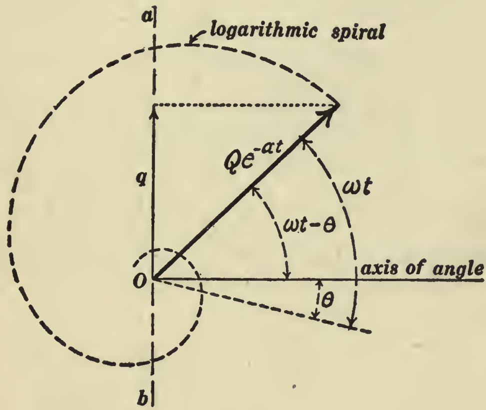


Fig. 87.

at a constant angular velocity of ω radians per second and grows continually shorter and shorter. The end of the line Qe^{-at} describes the dotted curve which is called an exponential spiral.

PROBLEMS.

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} + 4y = 0.$ Ans. $y = C \sin 2x + D \cos 2x.$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 0.$ Ans. $y = e^x(C \sin 2x + D \cos 2x).$

3. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 25y = 0.$ Ans. $y = e^{4x}(C \sin 3x + D \cos 3x).$

4. $\dfrac{d^2y}{dx^2} - 24\dfrac{dy}{dx} + 169y = 0.$ Ans. $y = e^{12x}(C \sin 5x + D \cos 5x).$

5. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$ Ans. $y=e^{-\frac{1}{2}x}\left(C\sin\frac{\sqrt{3}}{2}x+D\cos\frac{\sqrt{3}}{2}x\right).$

106. Forced or maintained oscillations.—The weight in Fig. 84 is acted upon by the force $-\frac{q}{C}$ due to the altered stretch of the spring, and by the force $-R\frac{dq}{dt}$ due to the frictional drag of the fluid in which the weight moves up and down. Let us suppose that an outside periodic force, $E\sin pt$, acts upon the weight. Then the total force acting on the weight will be $-\frac{q}{C}-R\frac{dq}{dt}+E\sin pt$, and this will be equal to the product of mass times acceleration $\left(L\frac{d^2q}{dt^2}\right)$. Therefore we will have:

$$q+CR\frac{dq}{dt}+LC\frac{d^2q}{dt^2}=CE\sin pt \tag{1}$$

This is the differential equation of motion of the weight under the assumed conditions, and this identical differential equation expresses the mode of variation of the charge q in the condenser C in Fig. 88 when an alternator of which the electromotive force is $E\sin pt$ delivers electric current to a circuit of resistance R and inductance L, the circuit containing a condenser of capacity C. The charge q in the condenser is not so familiar a thing as the current $i\left(=\frac{dq}{dt}\right)$ which flows in the circuit, but it is convenient to use q as our variable instead of i.

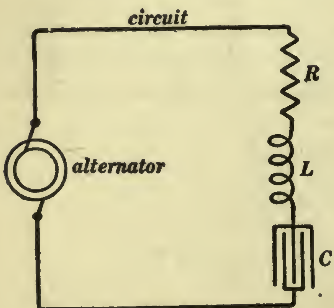


Fig. 88.

Everyone who has had any experience with alternating-current phenomena knows that in general there are two recognizable alternating currents in the circuit shown in Fig. 88, namely, (*a*) *The alternating current which is maintained by the alternator.* This alternating current is of the same frequency as the electromotive force $E \sin pt$ of the alternator, and (*b*) *A decaying oscillatory current, as if the condenser had been initially charged and allowed to discharge through the circuit.* This decaying alternating current is independent of the alternator, it is expressed by equation (2) of Art. 105, and its frequency is determined by equation (4) of Art. 105.

The finding of the general solution of equation (1) is accomplished in two steps. The first step is the finding of a so-called singular solution of (1), a solution which expresses the maintained alternating current above mentioned, and the second step is to add to this singular solution the general solution of equation (1) of Art. 105 as given by equation (2) of Art. 105. This addition is permissible inasmuch as it is very easy to show that $v + z$ is a solution of equation (1) of this article if v is a solution, and if z is a solution of equation (1) of Art. 105.

This procedure involves elaborate transformations which are unintelligible to the beginner, and it is therefore useless to carry it out. Fortunately, however, a much simpler method is available for the establishment of the desired result, a result which is the foundation of the more important part of the theory of alternating currents.*

* See pages 66–68 and 73–74 of Franklin and Esty's *Elements of Electrical Engineering*, Vol. II.

CHAPTER VIII.

THE PARTIAL DIFFERENTIAL EQUATION OF WAVE MOTION.

107. Differential equation of travel. Equation of a traveling curve.—When a point is stationary its abscissa x is constant, and when a point travels at constant velocity along the x-axis of reference its abscissa increases (or decreases) at a constant rate so that $x = kt + \textit{any constant}$. This is all very simple but the equation of a traveling curve which is so extensively used in the theory of wave motion is not so simple, and therefore some discussion of it is necessary.

The curve cc in Fig. 89 is stationary with respect to the origin O', and the equation of the curve referred to the origin O' is:

$$y = F(x') \tag{1}$$

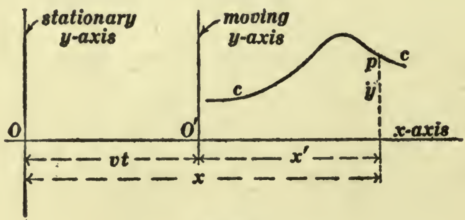


Fig. 89.

But the origin O' and the curve cc are both assumed to be traveling to the right at velocity v, so that the abscissa of the moving origin O' referred to the stationary origin O is vt. Therefore the abscissa x of any point on the curve cc referred to the stationary origin is equal to $x' + vt$, or $x' = x - vt$. Therefore, substituting $x - vt$ for x' in equation (1) we have

$$y = F(x - vt) \tag{2}$$

which is the equation of the traveling curve cc referred to the stationary origin O.

Similarly it may be shown that

$$y = f(x + vt) \tag{3}$$

is the equation of a curve which is traveling to the left at velocity v.

The two equations (2) and (3) satisfy one of the most important of the differential equations of physics, namely, the differential equation of wave motion.

Let the expression $x - vt$ be represented by the single letter z. That is

$$z = x - vt \tag{4}$$

so that

$$\frac{\partial z}{\partial x} = 1 \tag{5}$$

and

$$\frac{\partial z}{\partial t} = -v \tag{6}$$

Then equation (2) becomes

$$y = F(z) \tag{7}$$

Let $\dfrac{dy}{dz}$ be represented by $F'(z)$, and let $\dfrac{d^2y}{dz^2}\left[=\dfrac{d[F'(z)]}{dz}\right]$ be represented by $F''(z)$. Then, according to the rule for differentiating a function of a function (see Art. 34), we have (using the above values of $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial t}$):

$$\frac{\partial y}{\partial x} = \frac{dy}{dz} \cdot \frac{\partial z}{\partial x} = F'(z) \tag{8}$$

differentiating again we have:

$$\frac{\partial^2 y}{\partial x^2} = \frac{d[F'(z)]}{dz} \cdot \frac{\partial z}{\partial x} = F''(z) \tag{9}$$

Similarly we have:

$$\frac{\partial y}{\partial t} = \frac{dy}{dz} \cdot \frac{\partial z}{\partial t} = -vF'(z) \tag{10}$$

and

$$\frac{\partial^2 y}{\partial t^2} = -\frac{v\, d[F'(z)]}{dz} \cdot \frac{\partial z}{\partial t} = +\, v^2 F''(z) \tag{11}$$

Therefore, comparing equations (9) and (11), we have

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \tag{12}$$

and it may be shown, as above, that equation (3) leads to this same differential equation.

A partial differential equation may be recognized as one which contains more than one independent variable. Therefore it is unnecessary to use the symbol ∂ instead of the symbol d. Hereafter the symbol d will be used exclusively.

General solution of equation (12).—Equations (2) and (3) both satisfy equation (12), and therefore equations (2) and (3) are both solutions of (12). Therefore, according to the principle of superposition as explained in Art. 100, the sum of $F(x - vt)$ and $f(x + vt)$ or

$$y = F(x - vt) + f(x + vt) \tag{13}$$

is a solution of equation (12). Indeed this is the general solution of (12) because it contains two undetermined functions.*

PROBLEMS.

1. Find the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ when

$$y = A \sin (x - vt);$$

also when

$$y = A \sin (x + vt).$$

Show that both of these functions satisfy equation (12) of Art. 107.

* The general solution of an ordinary differential equation of the second order contains two undetermined constants, and the general solution of a partial differential equation of the second order contains two undetermined functions.

2. Show that

$$A \sin (x - vt) + A \sin (x + vt) = 2A \sin x \cos vt,$$

and plot the curve $y = 2A \sin x \cos vt$ for the following values of vt, namely,

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} \text{ and } 2\pi.$$

Note.—The curves thus plotted show the successive configurations of a string which is vibrating in what is called a simple mode. See Franklin and MacNutt, *Light and Sound*, page 243, The Macmillan Co., New York, 1909

108. Equation of motion of a stretched string.—When a stretched string is in equilibrium it is of course straight. Let us choose this equilibrium position of the string as the x-axis of reference as shown in Fig. 90, and let us set up the differential

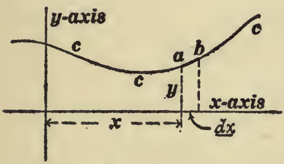


Fig. 90.

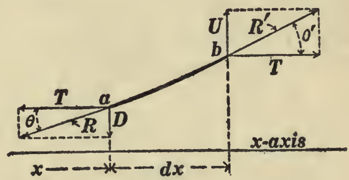


Fig. 91.

equation which expresses the mode of motion of the string while the string is vibrating or while a bend is traveling along the string as a wave. We will assume that each part of the string moves only up and down (at right angles to the x-axis in Fig. 90) and we will assume that the string is perfectly flexible.* Under these conditions the x-component of the tension of the string is always and everywhere equal to a constant T.

Let the curve ccc, Fig. 90 be the configuration of the string at a given instant t, that is, ccc is what a photographer would call a snapshot of the moving string. The shape of the curve ccc

* This means that the only thing that keeps the string straight is its tension.

defines y as a function of x, and the steepness of the curve at a point is the value of $\frac{dy}{dx}$ at that point.

Consider the very short portion ab of the string. The length of this portion of the string when the string lies along the x-axis (in equilibrium) is dx, and the mass of the portion is $m \cdot dx$ pounds or grams as the case may be, where m is the mass per unit length of the string. An enlarged view of the very short portion of the string is shown in Fig. 91. The adjacent parts of the string pull on the portion ab in the direction of the string at a and at b, and the forces R and R' are thus exerted on the portion ab. The x-component of R is the force T towards the left, and the x-component of R' is an equal force T towards the right. Therefore the downward force D on the end a of the portion ab is $D = T \tan \theta$, and the upward force U acting on the end b is $U = T \tan \theta'$. Therefore the net upward force exerted on ab is

$$dF = T \tan \theta' - T \tan \theta \tag{1}$$

But $\tan \theta$ is equal to the value of $\frac{dy}{dx}$ at a, and $\tan \theta'$ is equal to the value of $\frac{dy}{dx}$ at b. Therefore the difference, $\tan \theta' - \tan \theta$, is the increase of $\frac{dy}{dx}$ from a to b, and this increase is equal to $\frac{d^2y}{dx^2} \cdot dx$. This is evident when we consider that the value of $\frac{d^2y}{dx^2}$ means the rate of increase of $\frac{dy}{dx}$ with respect to x. Therefore, substituting $\frac{d^2y}{dx^2} \cdot dx$ for $\tan \theta' - \tan \theta$ in equation (1), we have:

$$dF = T \frac{d^2y}{dx^2} \cdot dx \tag{2}$$

Now according to Newton's laws of motion the net upward force dF acting on the portion ab of the string is equal to the

mass $m \cdot dx$ of the portion multiplied by the upward acceleration $\frac{d^2y}{dt^2}$ of the portion. Therefore, substituting $m\frac{d^2y}{dt^2} \cdot dx$ for dF in equation (2), we have:

$$m\frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2}$$

or

$$\frac{d^2y}{dt^2} = \frac{T}{m}\frac{d^2y}{dx^2} \tag{3}$$

This differential equation is here derived for the case of wave motion on a string, but identically the same form of equation applies to sound waves in air, to electric waves, to waves of light (which indeed are electric waves), and to certain kinds of water waves.* Equation (3) is therefore very important. Its general solution as found in Art. 107 is:

$$y = F(x - vt) + f(x + vt) \tag{4}$$

where

$$v = \sqrt{\frac{T}{m}} \tag{5}$$

The term $F(x - vt)$ represents a wave (a bend of any shape) traveling to the right at velocity v, and the term $f(x + vt)$ represents a wave (a bend of any shape) traveling to the left at velocity v.

109. Idea of wave motion established without the help of equation (3) of Art. 108.—It is evident from Art. 107 that *travel*, purely and simply, is about the only thing that is established by the solution of equation (3) of Art. 108. Therefore one might expect to obtain equations (4) and (5) without the help of equation (3) by introducing the idea of travel at the beginning. With this end in view let us consider a bend of any shape and let us imagine that this bend is traveling along the string to the right at any

* We are here considering only the simple case in which there are no appreciable energy losses as the wave travels along.

velocity v. One could make a bend of definite shape travel along a stretched string by threading the string through a bent tube and moving the bent tube along, and this state of affairs would be entirely unchanged if *we imagine the tube to be stationary and the stretched string to be drawn through it* as indicated in Fig. 92.

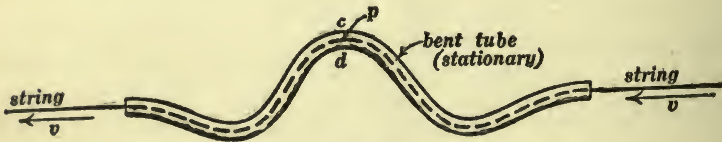


Fig. 92.

It is assumed that the string slides through the tube without friction. At any point p the string pushes against the side d of the tube because of its tension, and the string pushes outwards against the side c of the tube because of centrifugal action. Let r be the radius of curvature of the tube at p. Then the particles of the string near p may be considered to travel along a circular path of radius r. Therefore, according to Art. 50, the radial* acceleration of a particle of the string at p is $\frac{v^2}{r}$. Consider an infinitely short piece of the string whose length is ds and whose mass is $m \cdot ds$. Then $m \cdot ds \times \frac{v^2}{r}$ is the radial force which must act upon the short piece of string to produce the specified acceleration $\frac{v^2}{r}$. If the string has no tension, then all of this radial force is exerted by the side c of the tube.

If the string is under tension T, then the tension produces a radial force equal to $\frac{T}{r} \cdot ds$ on the portion ds of the string, according to Art. 51; and if the string is not moving this force is exerted against the side d of the tube.

* In the direction of r and towards the center of the osculating circle.

If the string is moving at a velocity which satisfies the equation:

$$m\frac{v^2}{r} = \frac{T}{r}$$

or

$$v = \sqrt{\frac{T}{m}} \tag{1}$$

then the radial force due to the tension of the string is just sufficient to produce the necessary radial acceleration, and no force at all is exerted on the string by the guiding tube. Under these conditions the guiding tube might be removed and the bend would stand in a fixed position on the traveling string; or if the string were standing still the bend would travel along the string at velocity v.

In this discussion the bend can be of any shape and it can travel at velocity v in either direction.

110. The vibration of a plucked string.*—A string is pulled to one side as shown in Fig. 93 and released. The string is thus

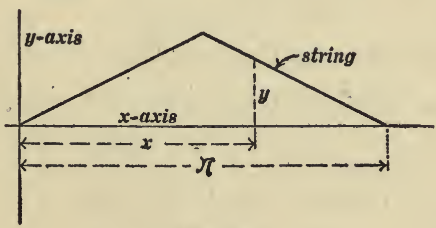


Fig. 93.

* The vibration of a string which is struck with a hammer as in a piano is easy to formulate. The vibration of a string which is set vibrating by a bow as in a violin is somewhat more difficult to formulate.

A good discussion of this subject is given in Byerly's *Fourier's Series and Spherical Harmonics*, pages 30–145, Ginn and Co., Boston, 1893.

A discussion of the motion of piano strings and of violin strings is given in Appendices V and VI of Helmholtz's *Tonempfindungen*. English translation by Alexander J. Ellis is entitled *Sensations of tone*. Published by Longmans, Green and Co., 1885.

set vibrating. The vibrating string has at each instant a definite shape, that is, it forms a definite curve. It is desired to find an equation expressing y in terms of x and elapsed time t, which equation will be at each instant the equation of the curve formed by the moving string.

There are two conditions which must be satisfied by the expression for y, namely: (a) y must be zero at all times when $x = 0$ and when $x = \pi$, and (b) at the instant $t = 0$ the expression for y must be the equation of the curve formed by the plucked string at the moment of release.

Also, of course, the expression for y must satisfy the differential equation (3) of Art. 108, namely,

$$\frac{d^2y}{dt^2} = \frac{T}{m}\frac{d^2y}{dx^2} \tag{1}$$

Now it can be shown (see problem 2 on page 193) that the following expressions satisfy equation (1)

$$y = A \sin nx \sin nvt \tag{2}$$
$$y = A \sin nx \cos nvt \tag{3}$$
$$y = A \cos nx \sin nvt \tag{4}$$
$$y = A \cos nx \cos nvt \tag{5}$$

where A and n have any values whatever. But only (2) and (3) give $y = 0$ when $x = 0$, and n *must be an integer* to give $y = 0$ when $x = \pi$. But (2) cannot be used because it gives $y = 0$ everywhere when $t = 0$. Therefore our problem is to take

$$y = A \sin nx \cos nvt \tag{3}$$

which satisfies condition (a), and add a large number of such solutions together (using different values of A and n in each) to get an expression for y which satisfies condition (b), above. The possibility of doing this was discovered by Fourier* and is embodied in what is called Fourier's theorem.

* Fourier's original discussion is very simple and interesting. See Fourier's *Theory of Heat*, translated by Alexander Freeman, Cambridge, 1878.

111. Fourier's theorem.—Let y be any given function of x. Consider a certain portion AB of the curve which represents y as shown in Fig. 94, and let the distance AB be 2π (this is equivalent to choosing $\frac{AB}{2\pi}$ as our unit of length). Then, according to Fourier we may express the function y *within the region* AB as follows:

$$\left.\begin{aligned} y = A_0 + A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots \\ + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x + \cdots \end{aligned}\right\} \tag{1}$$

where the coefficients $A_0, A_1, A_2, \cdots$ and $B_1, B_2, B_3, \cdots$ are constants whose values are:

$$A_0 = \frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \cdot dx \tag{2}$$

$$A_n = \frac{1}{\pi}\int_{x=0}^{x=2\pi} y \sin nx \cdot dx \tag{3}$$

$$B_n = \frac{1}{\pi}\int_{x=0}^{x=2\pi} y \cos nx \cdot dx \tag{4}$$

A complete proof of Fourier's theorem would involve a proof that the right-hand member of equation (1) is a convergent

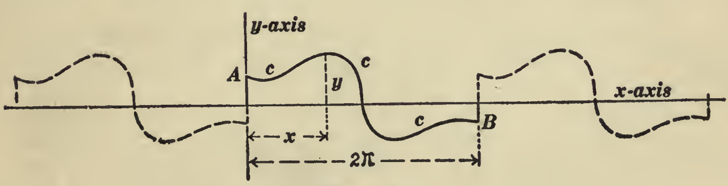


Fig. 94.

series.* This, however, will here be taken for granted. The proof then reduces to a derivation of equations (2), (3) and (4).

* This matter is discussed in Chapter III of Byerly's *Fourier's Series and Spherical Harmonics*, Ginn and Co., Boston, 1893.

In this proof a consideration of average values is most important, and the student should therefore look over Art. 73 again.

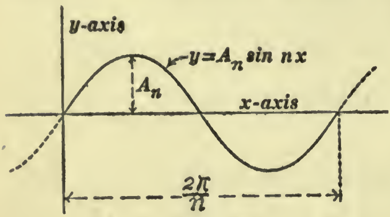


Fig. 95.

Another matter of importance is to understand clearly the meaning of the equation $y = \sin nx$ (and $y = \cos nx$) when n is an integer. Now $y = \sin nx$ is the equation of a sine curve. If $n = 1$ this sine curve makes a positive "arch" between $x = 0$ and $x = \pi$, and a negative "arch" between $x = \pi$ and $x = 2\pi$. In general the sine curve $y = \sin nx$ makes a positive arch between $x = 0$ and $x = \frac{\pi}{n}$, and a negative arch between $x = \frac{\pi}{n}$ and $x = \frac{2\pi}{n}$ as shown in Fig. 95. Therefore the sine curve $y = \sin nx$ makes n pairs of positive and negative arches between $x = 0$ and $x = 2\pi$.

The derivation of equations (2), (3) and (4) depends upon the following propositions, n and m being integers.

Proposition I.—*The average value of* $\sin nx$ (*or of* $\cos nx$) *between* $x = 0$ *and* $x = 2\pi$ *is zero.* This is evident when we consider that $\sin nx$ (or $\cos nx$) passes through exactly similar sets of positive and negative values which exactly offset each other between $x = 0$ and $x = 2\pi$.

Proposition II.—*The average value of* $\sin^2 nx$ (*or of* $\cos^2 nx$) *between* $x = 0$ *and* $x = 2\pi$ *is equal to* $\frac{1}{2}$. This may be shown by finding the value of $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} \sin^2 nx \cdot dx$ (or of the corresponding expression for $\cos^2 nx$) as explained in Art. 73.

Proposition III.—*The average value of* $\sin nx \sin mx$ *between* $x = 0$ *and* $x = 2\pi$ *is zero when* n *and* m *are different integers.*

This may be shown by finding the value of the expression

$$\frac{1}{2\pi}\int_{x=0}^{x=2\pi} \sin nx \sin mx \cdot dx,$$

which may be easily done with the help of form 24 in the table of integrals.

Proposition IV.—*The average value of* $\sin nx \cos mx$ *between* $x = 0$ *and* $x = 2\pi$ *is zero whether the integers* n *and* m *be the same or not.* This may be shown with the help of form 25 in the table of integrals.

Proposition V.—*The average value of* $\cos nx \cos mx$ *from* $x = 0$ *to* $x = 2\pi$ *is zero when* m *and* n *are different integers.* This can be shown with the help of form 26 in the table of integrals.

Derivation of equation (2).—Consider the average value of each member of equation (1) between $x = 0$ and $x = 2\pi$. The average value of the first member is, by definition, equal to $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \cdot dx$, and the average value of the second member is A_0 according to proposition I above. Therefore $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \cdot dx = A_0$ which is equation (2).

Derivation of equation (3).—Multiply both members of equation (1) by $\sin nx$ and we have:

$$y \sin nx = A_0 \sin nx + A_1 \sin nx \sin x + \cdots + A_n \sin^2 nx \\ + \cdots + B_1 \sin nx \cos x + \cdots.$$

Consider the average value of each member of this equation between $x = 0$ and $x = 2\pi$. The average value of the first member is $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \sin nx \cdot dx$. The average value of every term of the second member is zero according to the above propositions except the term $A_n \sin^2 nx$, and the average value of this term

is $\frac{1}{2}A_n$ according to proposition II. Therefore

$$\frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \sin nx \cdot dx = \tfrac{1}{2}A_n$$

which is equation (3)

Derivation of equation (4).—Multiply both members of equation (1) by $\cos nx$, consider the average value of each member of the resulting equation between $x = 0$ and $x = 2\pi$, and equation (4) is obtained.

Simplification of equations (3) and (4) due to symmetry of given curve.—Let ccc, Fig. 96a, be the curve which is to be expressed by equation (1). It is permissible to take the base of ccc as π.

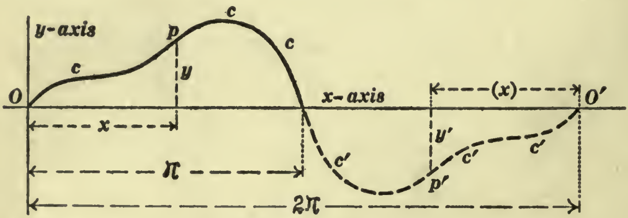


Fig. 96a.

Under these conditions there are two important methods for choosing the second half of the complete curve, namely, the portion between π and 2π, as follows:

Curves having sine terms only.—The second half of the complete curve may be chosen like ccc turned upside down and turned end for end, as shown by the dotted curve $c'c'$ in Fig. 96a. In this case $y = -y'$ as shown in Fig. 96a. Now $\sin nx$ has equal and opposite values at p and at p', therefore $y \sin nx \cdot dx$ has the same value and sign at p and at p', and therefore

$$\int_0^{2\pi} y \sin nx \cdot dx = 2\int_0^{\pi} y \sin nx \cdot dx$$

Consequently equation (3) becomes:

$$A_n = \frac{2}{\pi}\int_0^\pi y \sin nx \cdot dx \tag{5}$$

Furthermore $\cos nx$ has the same value and the same sign at p and at p', therefore $y \cos nx \cdot dx$ has the same value but has opposite signs at p and at p', and therefore $\int_0^{2\pi} y \cos nx \cdot dx$ is zero. Consequently all of the cosine terms drop out of equation (1). Furthermore it is evident from the symmetry of the complete curve $cccc'c'$ in Fig. 96*a* that $\int_0^{2\pi} y \cdot dx = 0$ so that $A_0 = 0$. Therefore the complete curve in Fig. 96*a* is given by:

$$y = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots \tag{6}$$

Curves having cosine terms only.—The second half of the complete curve may be chosen like ccc turned end for end but not turned upside down as shown by the dotted curve $c'c'c'$ in Fig. 96*b*.

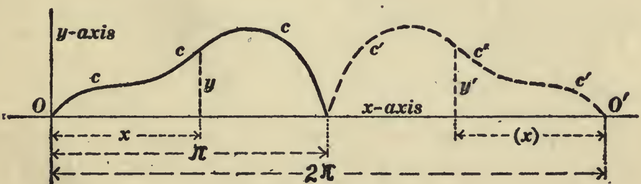


Fig. 96*b*.

Under these conditions it may be shown, by an argument somewhat similar to the above, that equations (1) and (4), respectively, become

$$y = A_0 + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x + \cdots \tag{7}$$

and

$$B_n = \frac{2}{\pi}\int_0^\pi y \cos nx \cdot dx \tag{8}$$

112. The vibration of a plucked string completely formulated.— In Art. 110 the problem of the plucked string was reduced to the problem of adding together a number of terms like $A \sin nx \cos nvt$ so as to get an expression for y which is the equation of the curve formed by the string when $t = 0$. This is evidently the same thing as expanding the given function y in a series of sines as explained in Art. 111. That is, we must find the coefficients in the series:

$$y = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots \tag{1}$$

and the general expression for the coefficient A_n is:

$$A_n = \frac{2}{\pi}\int_0^\pi y \sin nx \cdot dx \tag{2}$$

according to equation (5) of Art. 111, the use of which means that we have assumed the length of our string as π units and

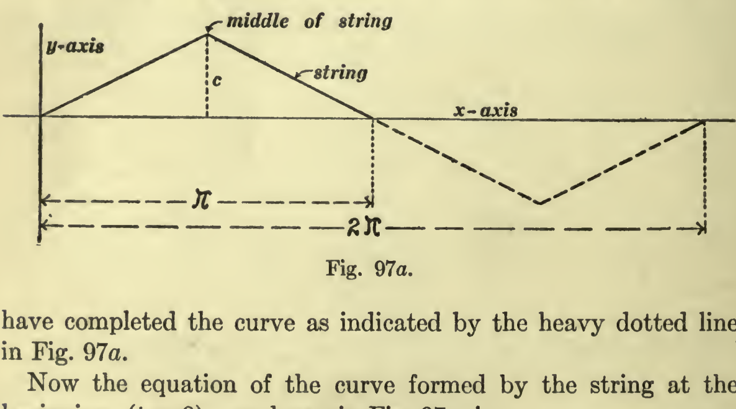


Fig. 97*a*.

have completed the curve as indicated by the heavy dotted line in Fig. 97*a*.

Now the equation of the curve formed by the string at the beginning $(t = 0)$, as shown in Fig. 97*a*, is:

$$y = \frac{2c}{\pi}x \quad \text{from} \quad x = 0 \quad \text{to} \quad x = \frac{\pi}{2} \tag{3}$$

$$y = 2c - \frac{2c}{\pi}x \quad \text{from} \quad x = \frac{\pi}{2} \quad \text{to} \quad x = \pi \tag{4}$$

Therefore the integral (2) must be evaluated in two parts, namely:

$$A_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{2c}{\pi} x \sin nx \cdot dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \left(2c - \frac{2c}{\pi} x\right) \sin nx \cdot dx \qquad (5)$$

which gives:

$$A_n = \frac{8c}{\pi^2}\frac{1}{n^2} \sin \frac{n\pi}{2}$$

Therefore the equation of the curve formed by the string at the beginning is:

$$y = \frac{8c}{\pi^2}(\sin x - \tfrac{1}{9} \sin 3x + \tfrac{1}{25} \sin 5x - \cdots) \qquad (6)$$

and the equation of the curve formed by the string at an instant t seconds after the beginning is:

$$y = \frac{8c}{\pi^2}(\sin x \cos vt - \tfrac{1}{9} \sin 3x \cos 3vt + \cdots) \qquad (7)$$

Description of the motion of a plucked string.—It would seem from the elaborate difficulties of the above problem [and perhaps

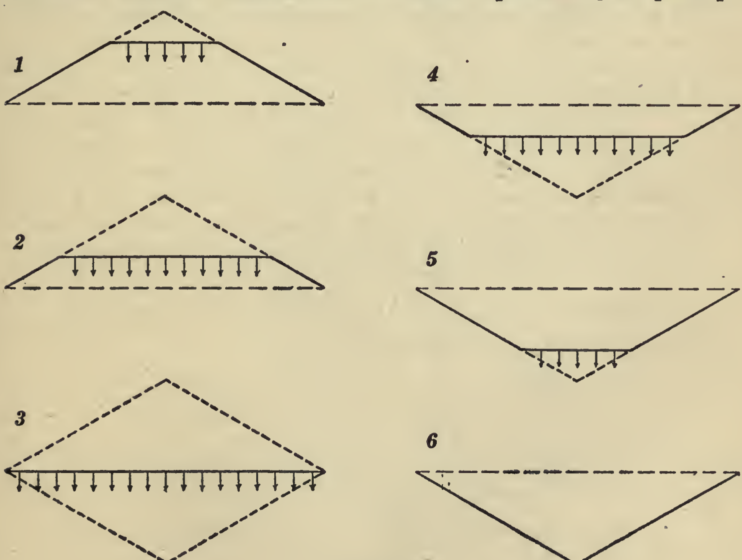


Fig. 97b.

the complexity of the result as given in equation (6) would suggest] that the motion of a plucked string is very complicated. As a matter of fact, however, the motion of a plucked string is extremely simple and easy to describe. Thus Fig. 97*b* shows six successive snapshots of a string which has been plucked in the middle and released. The moving part of the string is straight and parallel to the equilibrium position of the string and the motion is indicated by the short arrows.

PROBLEMS.

The most important practical problem involving Fourier's theorem is, perhaps, the finding of the coefficients in equation (1) of Art. 111 when the function y is given not as an algebraic function of x but as an experimentally determined curve.

A very good set of directions for making the necessary calculations is given in Bedell and Pierce's *Direct and Alternating Current Manual*, 2nd edition, pages 331–338, D. Van Nostrand, New York, 1913; and a discussion of the origin and proof of the method is given on pages 339–344.

The harmonic analyzer is a machine for finding the coefficients in equation (1) of Art. 111 when the function y is given as an experimentally determined curve. The earliest harmonic analyzer is that of Lord Kelvin. A brief description of this machine is given in Franklin's *Electric Waves*, pages 340–342, The Macmillan Co., New York, 1909. See also section 37 of the article on *Tides* in the ninth edition of the *Encyclopedia Britannica*. See also articles by James Thomson and by Sir Wm. Thomson (Lord Kelvin) in *Proceedings of the Royal Society*, Vol. XXIV, 1876, page 262 and pages 269 and 271. The harmonic analyzer of G. U. Yule is described in an article by J. N. Le Conte, *Physical Review*, Vol. VII, pages 27–34, 1898. An especially interesting description of a harmonic analyzer is given on pages 68–74 of A. A. Michelson's *Light waves and their uses*, University of Chicago Press, Chicago, 1903.

1. Determine the coefficients in Fourier's series to give the curve cc which is shown in Fig. $p1$.

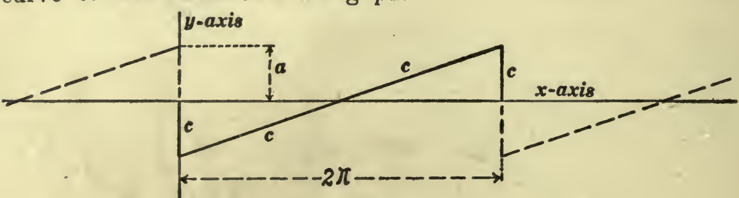


Fig. $p1$.

Ans: $y = -\frac{2a}{\pi}(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots)$

2. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. *p*2.

Ans: $y = \frac{8a}{\pi^2}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$

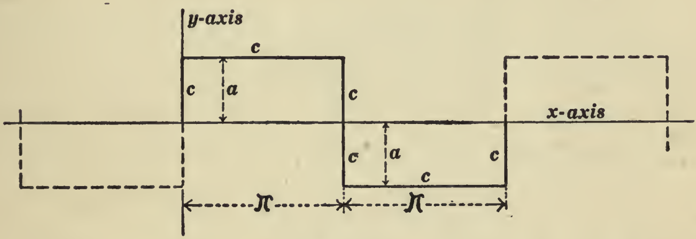


Fig. *p*2.

3. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. *p*3.

Ans: $y = \frac{8a}{\pi^2}(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots)$

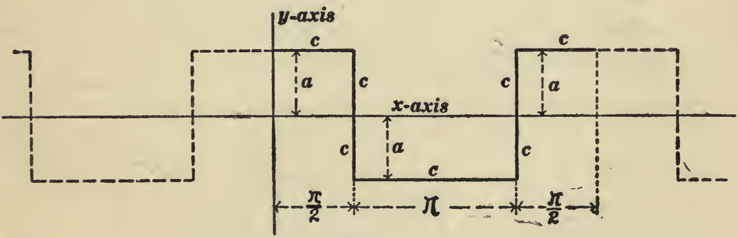


Fig. *p*3.

4. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. *p*4.

Ans: $y = \frac{8a}{\pi^2}(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \frac{1}{49}\sin 7x + \cdots)$

5. Determine the coefficients in Fourier's series to give the curve cc which is shown in Fig. $p5$.

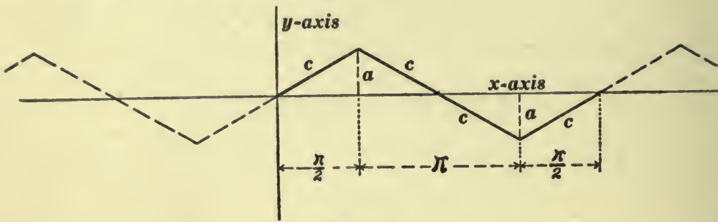


Fig. $p4$.

Ans: $y = \frac{8a}{\pi^2}(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots)$

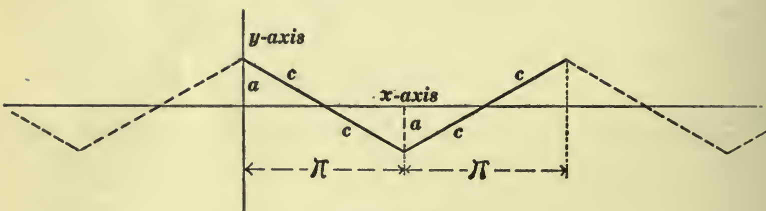


Fig. $p5$.

6. Determine the coefficients in Fourier's series to give the curve cc which is shown in Fig. $p6$.

Ans: $y = \frac{2a}{\pi}(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots)$

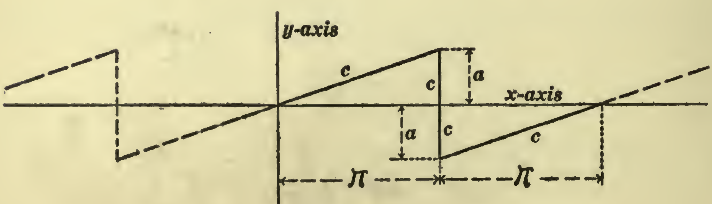


Fig. $p6$.

7. Derive the answer to problem 4 by integrating the answer to problem 3.

Note.—Any Fourier series gives a convergent series when integrated term by term.

8. Plot the curve which represents the integral of the curve given in Fig. $p2$, the constant of integration being such as to make the integral curve pass through the origin; and determine the coefficients in a Fourier's series to give the integral curve.

Note.—The constant of integration is zero in problem 7 and therefore easily overlooked.

9. Derive the answer to problem 3 from the answer to problem 2 by shifting the origin $\frac{\pi}{2}$ to the right.

10. Differentiate the answer to problem 4 and interpret the result.

Note.—A Fourier series gives a divergent series by differentiation unless the function y is continuous. Thus y in Fig. $p4$ is continuous, whereas y in Figs. $p1$, $p2$, $p3$ and $p6$ is discontinuous.

CHAPTER IX

VECTOR ANALYSIS.

SCALAR AND VECTOR FIELDS.

113. Space analysis.—A system of *space analysis,* commonly called *vector analysis,* is developed in this chapter, and it is extremely useful and important in every branch of physics where variations in space are involved. Thus the theory of electricity and magnetism (aside from the electron theory) is a simple application of vector analysis. Also the theory of heat flow, the theory of fluid motion, and a great part of the theory of elasticity and wave motion are applications of vector analysis. It is almost impossible for a student to make a beginning in any branch of theoretical physics without some understanding of vector analysis.

Vector analysis as outlined in this chapter is a very different thing from the familiar use of vectors and of complex quantity in the theory of alternating currents. In the one case we have a comprehensive system of space analysis, and in the other case we have a narrow scheme for representing the solution of a linear partial differential equation, as explained in chapter VII. Indeed the kind of "vector analysis" which is used in the theory of alternating currents cannot possibly be extended to three dimensions as a consistent system of space analysis.

Vector analysis originated in Sir William Hamilton's theory of quaternions.* The theory of quaternions, however, contains much that is of doubtful value in theoretical physics. Indeed Maxwell in his great treatise on *Electricity and Magnetism* (Oxford, 1873) used only a few of Hamilton's ideas. The study of Max-

* See Sir William Rowan Hamilton's *Lectures on Quaternions*, and *Elements of Quaternions*. These books are now out of print but they may be found in most good libraries. *An Elementary Treatise on Quaternions* by P. G. Tait, second edition, Oxford, 1873, is the standard treatise on the subject.

well's treatise has kept the subject of space analysis before the serious student of physics for more than a generation, and two significant attempts have been made to formulate the more useful of Hamilton's ideas in what is now called vector analysis. The first of these was that of Willard Gibbs, whose view of vector analysis was outlined in a very condensed form in a pamphlet printed (for private circulation, only) in New Haven in 1883.* The second attempt to formulate the more useful of Hamilton's ideas was that of Oliver Heaviside.†

The first attempt to place vector analysis before the student of physics in simple form was that of E. L. Nichols and W. S. Franklin.‡ The best discussion of vector analysis for the student is that of Abraham and Föppl in their *Theorie der Electricität*, Vol. I, pages 1–125, Leipsig, 1907.

114. Scalar and vector quantities.—A scalar quantity is a quantity which has magnitude only. Thus every one recognizes at once that to specify 10 cubic yards of sand, 25 pounds of sugar, 5 hours of time is in each case to make a complete specification. Such quantities as volume, mass, time and energy are scalar quantities.

A vector quantity is a quantity which has both magnitude and direction, and to completely specify a vector quantity one must give both its magnitude and its direction. This necessity of specifying both the magnitude and the direction of a vector is especially evident when one is concerned with the relationship of two or more vectors. Thus if a man travels a stretch of 10 miles and then a stretch of 5 miles more, he is by no means necessarily 15 miles from home. If one man pulls on a post with a force of

* Professor Gibbs' point of view is set forth in *Vector Analysis* by E. B. Wilson, New Haven, Yale University Press, 1901.

† See Heaviside's *Electromagnetic Theory*, Vol. I, pages 132–305, The Electrician Publishing Co., London, 1893. This discussion of Heaviside's is unusually interesting, but we cannot agree with Heaviside in his statement that vector analysis is independent of the quaternion.

‡ See Nichols and Franklin's *Elements of Physics*, Vol. II, first edition, The Macmillan Co., New York, 1895.

200 units and another man with a force of 100 units, the total force acting on the post is by no means necessarily equal to 300 units. A New Yorker traveling steadily at a speed of 60 miles an hour would by no means necessarily reach Boston in 5 hours, because he might be traveling in some other direction.

Many cases arise in physics where it is necessary to consider the single force which is equivalent to the combined action of several given forces; where it is necessary to consider the actual velocity of a body due to the combined action of several causes, each of which alone would produce a certain amount of velocity in a given direction; and so on. The single force or single velocity is in each case called the *vector sum* or the *resultant* of the given forces or given velocities.

The addition of vector quantities is exemplified by the addition of forces as follows: Two given forces are represented by the lines a and b, in Fig. 98, and the sum or resultant of the forces is represented by the diagonal r of the parallelogram constructed on a and b as sides. It is evident that the geometrical relation

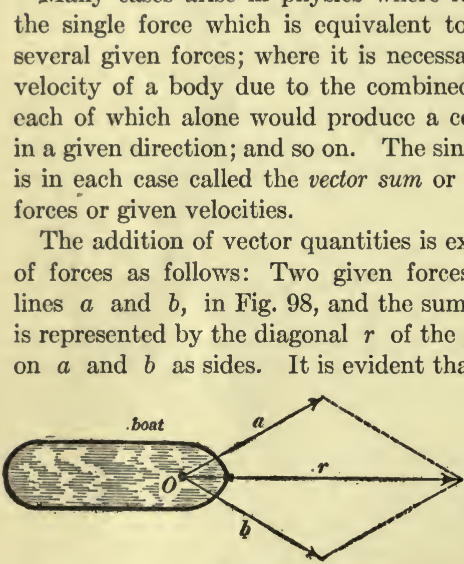


Fig. 98.

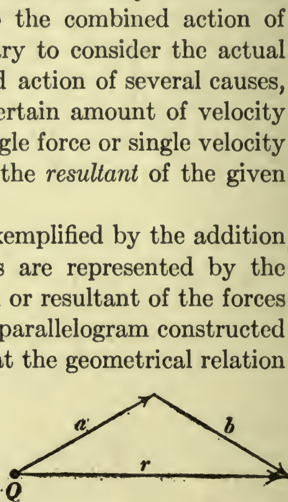


Fig. 99.

between a, b and r is completely shown by the triangle in Fig. 99, in which the line which represents force b is drawn from the extremity of the line which represents force a, and r is the closing line of the triangle.

The geometrical construction of Fig. 99 gives a method of adding any number of forces with the least amount of complication, as shown in Fig. 100. Thus to add four given forces a, b, c and d: draw a line representing force a; from the end of this line draw a line representing force b; from the end of this

line draw a line representing force c; and so on. Then the closing line r of the polygon $abcd$ represents the sum of the forces in magnitude and in direction.

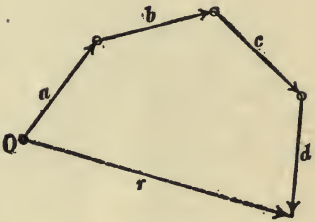


Fig. 100.

To represent a vector in an algebraic discussion it is usually most convenient to represent the vector in terms of its components in the directions of three chosen rectangular axes of reference. Thus if X, Y, Z are the components of a given force F (or any vector whatever) then the force may be represented as

$$F = X + Y + Z.$$

In this expression vector addition is of course understood because X, Y and Z are vectors.

115. Vector products. *Case I. Parallel vectors.*—The product of two parallel vectors is a scalar. This fact is exemplified in every branch of physics. Thus to multiply a force by the distance a body has moved in the direction of the force gives the work done by the force, and work is a scalar quantity. To multiply a force by the velocity with which a body moves in the direction of the force gives the power developed by the force, and power is a scalar quantity. The square of the velocity of a body determines its kinetic energy, and energy is a scalar quantity. A plane area is a vector quantity and its vector direction is the direction of the normal to the area. To multiply the area of the base of a prism by the altitude of a prism gives the volume of the prism, and volume is a scalar quantity.

The quotient of two parallel vectors is a scalar quantity. Thus let F be the force exerted by a fluid on a flat surface of area a. Then the quotient $\frac{F}{a}$ is the hydrostatic pressure of the fluid, and hydrostatic pressure is a scalar quantity.

Case II. Orthogonal vectors.—Vectors which are at right angles to each other are said to be *orthogonal.* The product of two orthogonal vectors is a third vector at right angles to both of the given vectors. Thus to multiply the length by the breadth of a rectangle gives the area of the rectangle, and area is a vector as above explained. To multiply a force F by the perpendicular distance l from the line of action of the force to a chosen axis gives the torque action lF of the force about the axis, and torque action is a vector quantity.

The quotient of two orthogonal vectors is a third vector at right angles to both of the given vectors. Thus to divide the area of a rectangle by the length of one side gives the length of the other side.

Case III. Oblique vectors.—The product of two oblique vectors consists of two parts; one part is a scalar and the other part is a vector. This proposition is a necessary result of the above statements concerning the product of parallel vectors and the product of orthogonal vectors. Thus Fig. 101 shows two oblique vectors U and V. The vector V can be resolved into the components V' and V'' parallel to U and perpendicular to U respectively, and then $V' + V''$ can be substituted for V in the product UV, giving $UV' + UV''$. But UV' is the product of two parallel vectors and it is a scalar, and UV'' is the product of two orthogonal vectors and it is a vector.

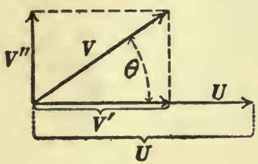


Fig. 101.

From Fig. 101 it is evident that $V' = V \cos\theta$ and that $V'' = V \sin\theta$. Therefore we have the following important propositions:

(*a*) The scalar part of the product of two oblique vectors is equal to the product of the numerical values of the respective vectors and the cosine of the angle between them.

(*b*) The numerical value of the vector part of the product of two oblique vectors is equal to the product of the numerical values

of the respective vectors and the sine of the angle between them; and of course the direction of the vector part of the product is at right angles to the plane of the given oblique vectors.

There are but few cases in physics where the *scalar part* and the *vector part* of the product of two oblique vectors are *both* important, although there are many cases where the *scalar part* of a vector product is important, and many other cases where the *vector part* of a vector product is important. Thus a force F acts on a car as shown in Fig. 102. Imagine the car to move a distance l in

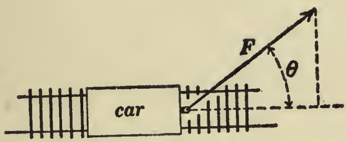


Fig. 102.

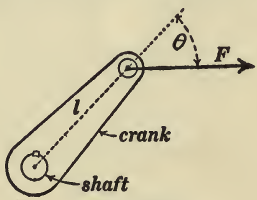


Fig. 103.

the direction of the track, then the work done by the force is the scalar part of the product of the two oblique vectors F and l, or: the work done is equal to *numerical value of* $l \times$ *numerical value of* $F \times \cos\theta$. The vector part of the product lF in this case has no very important meaning.

A force F acts on a crank as shown in Fig. 103. The torque action of the force about the axis of the crankshaft is the vector part of the product of the two oblique vectors F and l; the numerical value of the torque is equal to *numerical value of* $l \times$ *numerical value of* $F \times \sin\theta$, and the direction of the torque, considered as a vector, is parallel to the axis of the shaft.

Exact direction of the vector part of a vector product.—The product UV'' in Fig. 101 is a vector at right angles to the plane of the paper, but is it towards the reader or away from the reader? Consistency* requires us to admit that if the product UV'' is

* This is shown by the following discussion of Fig. 104 and by the interpretation of equation (1). Sir William R. Hamilton gives a very full discussion of this in his *Lectures* and in his *Elements of Quaternions*.

towards the reader in Fig. 101, then $V''U$ must be *away from* the reader. That is, we must admit that $UV'' = -V''U$ where U and V'' are orthogonal vectors.

This matter is exemplified as follows: A force F acts upon a crank-arm l as shown in Fig. 104. The vector part of the product lF is the torque action of the force about the axis O (perpendicular to the plane of the paper). Now the torque action of F in Fig. 104 can be expressed in terms of the components of F, namely, X and Y, and the components of l, namely, x and y. Thus xY is a counter-clockwise torque about O, and yX is a clockwise torque about O in Fig. 104. Therefore, taking counter-clockwise torques as positive, we have:

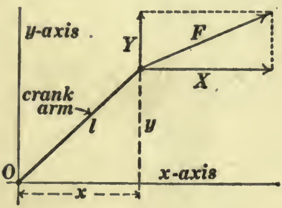


Fig. 104.

$$\left.\begin{array}{r}\text{Total torque action of } F\\ \text{about } O \text{ in Fig. 104, or the}\\ \text{vector part of the product } lF\end{array}\right\} = xY - yX \tag{1}$$

The vector direction of a torque may be thought of as the direction (along the axis of the torque) in which a right-handed screw would travel if turned by the torque. Adopting this convention and choosing the positive direction of the z-axis of reference towards the reader in Fig. 104 we have from equation (1):

(*a*) xY is an x-vector multiplied by a y-vector, and it is a vector in the positive direction of the z-axis.

(*b*) yX is a y-vector multiplied by an x-vector, and it is a vector in the negative direction of the z-axis.

Expressing F and l in terms of their components as shown in Fig. 104, we have:

$$F = X + Y \tag{2}$$

and

$$l = x + y \tag{3}$$

Multiplying equations (2) and (3) member by member, using the

ordinary rules of algebra, we get:

$$lF = xX + yY + xY + yX \tag{4}$$

But it is shown above that xY and yX are opposite in sign. Now the opposite signs of xY and yX are shown in equation (4), according to the above discussion, *by the fact that in one case we have an x-vector multiplied by a y-vector and in the other case we have a y-vector multiplied by an x-vector.* But it is very inconvenient to have algebraic signs so indirectly indicated; it is better to indicate the sign explicitly and take $xY - yX$ as the vector part of the product lF. *When this is done we pay no attention to the order of the factors, because the inverse order in the two terms is taken account of in the negative sign of the second term.* With this understanding, therefore, we have the following important propositions:

(*a*) The scalar part of the product lF in Fig. 104 is equal to $xX + yY$. This is evident when we consider that xX and yY are the only scalar terms in equation (4).

(*b*) The vector part of the product of lF in Fig. 104 is equal to $xY - yX$, and it is in the direction towards the reader in Fig. 104.

116. Scalar fields.—It is sometimes important to consider the temperature at various points in a body, the pressure at various points in a fluid, the density at various points of a substance or the electric charge per unit volume at various points in a region. Such a distribution of temperature, hydrostatic pressure, or density is called a *scalar field* because the quantity under consideration is a scalar and it has a definite value at each point in a region or field of space. The distribution is said to be *homogeneous* or *uniform* when the scalar quantity has the same value at every point in the field; otherwise the distribution is said to be *non-homogeneous.* An example of a non-homogeneous scalar field is the temperature of an iron rod one end of which is red hot and the other end of which is cold. Atmospheric pressure is also non-homogeneously distributed because it is different at different places and different at different altitudes.

A scalar field is sometimes called a *distributed scalar*. Thus when the scalar nature of temperature is to be emphasized it is helpful to call temperature (throughout a hot iron rod, for example) a distributed scalar.

A scalar field is sometimes called a *scalar function*. A distributed scalar has a definite value at each point of space; that is, the value of a distributed scalar at a point is a function of the coördinates of the point. When this fact is to be emphasized it is helpful to call temperature (or any distributed scalar) a scalar function.

117. Vector fields.—It is sometimes important to consider the velocity at different points of a fluid, the direction and intensity of heat flow at different points of a substance, or the direction and intensity of an electric or magnetic field at different points in space. Such a distribution of fluid velocity, or other vector, is called a *vector field* because the quantity under consideration is a vector, and it has a definite value and direction at each point in a region or field of space. The distribution is said to be *homogeneous* or *uniform* when the vector has the *same value* and is in the *same direction* at every point in the field; otherwise the distribution is said to be *non-homogeneous*. The water in a rotating bowl is an example of a non-homogeneous vector field because the water moves in different directions and at different velocities at different points in the bowl. The magnetic field around an electric wire is a non-homogeneous vector field because the magnetic field does not have the same intensity and the same direction everywhere.

A vector field is sometimes called a *distributed vector*. Thus when the vector nature of fluid velocity is to be emphasized it is helpful to call fluid velocity a distributed vector.

A vector field is sometimes called a *vector function*. Each component (the x-component, the y-component and the z-component) of the velocity of a fluid has a definite value at each point of space; that is, the values of the components at a point p are

functions of the coördinates of p. When this fact is to be emphasized it is helpful to call fluid velocity (or any distributed vector) a vector function.

118. Volume integral of a distributed scalar.—For the sake of simplicity let us consider a special case, namely, the density of a substance, and let us assume that the density varies from point to point. Let ψ be the density at a given point, then $\psi \cdot d\tau$ is the mass of material in the volume element $d\tau$ at the point, and the total mass M of the body is:

$$M = \int \psi \cdot d\tau \tag{1}$$

This integral is called the *volume integral of the distributed scalar* ψ. The physical significance of volume integral is not in every case so simple as in the case of density. If ψ is the volume density of electric charge, then equation (1) gives the total electric charge in the region throughout which the integration is extended. If ψ is the energy density in an electric or magnetic field, or in a strained solid, or in a moving fluid, then equation (1) gives the total energy in the region throughout which the integration is extended.

119. Gradient of a distributed scalar.—Any distributed scalar like temperature, or density, or hydrostatic pressure has a definite value at each point of a field and therefore the value of any distributed scalar at a point may be thought of as a function of the coördinates x, y and z of the point. Indeed, this matter has already been discussed in Arts. 62 to 66, where it is explained that a distributed scalar has a definite gradient at each point. Thus if ψ is the temperature at a point in a body, then

$$X = \frac{d\psi}{dx} \tag{1}$$

$$Y = \frac{d\psi}{dy} \tag{2}$$

$$Z = \frac{d\psi}{dz} \tag{3}$$

where X, Y and Z are the component gradients of ψ. The gradient of a distributed scalar has a definite value and a definite direction at each point and it is, therefore a distributed vector.

PROBLEMS.

1. A vessel one meter wide, one meter long, and one meter deep, contains a fluid of which the density is one gram per cubic centimeter at the top, increasing uniformly to two grams per cubic centimeter at the bottom. Find the volume integral of the density of the fluid. Ans. 1,500,000 grams.

2. The gradient of the density in problem 1 is uniform throughout the vessel and equal to one gram per cubic centimeter per meter, and it is directed vertically downwards. Suppose the downward gradient of the density to be a linear function of the distance from the top of the vessel, changing from zero at the top to two grams per cubic centimeter per meter at the bottom. Find the volume integral of the density of the fluid, the density being one gram per cubic centimeter at the top. Ans. 1,333,333 grams.

3. The value of a distributed scalar at any point p is $\psi = \frac{Q}{r}$ where Q is a constant and r the distance of p from the origin of coördinates. Find the x, y and z components of the gradient of ψ. Ans. The x-component is $-\frac{Qx}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.

4. Find the gradient of $\psi\left(=\frac{Q}{r}\right)$ in the direction of r. Ans. $-\frac{Q}{r^2}$.

5. The x-component of the gradient of any distributed scalar may be thought of as a distributed scalar. Find the y-component of the gradient of the x-component of the gradient of $\frac{Q}{r}$. Ans. $\frac{3Qxy}{(x^2+y^2+z^2)^{\frac{5}{2}}}$.

120. Permanent and varying states of scalar distribution.— When the temperature at each point of a body remains unchanged, the distribution of temperature throughout the body is said to be permanent, although, of course, the temperature of the body may not be everywhere the same. When the temperature at each point of a body is changing we have what is called a varying state of distribution. Thus the density in a gas which is being compressed, and the temperature throughout a body which is being heated or cooled are examples of varying scalar distributions.

121. Stream lines of a distributed vector.—A line drawn through a moving fluid so as to be at each point in the direction in which the fluid at the point is moving is called a *stream line.* The geometrical idea of a stream line applies to any vector field whatever, and the manner of distribution of a vector is clearly represented by the use in imagination of such lines. In electric and magnetic fields these lines are called *lines of force,* but the term *stream line* will be used in general statements.

122. Permanent and varying states of vector distribution.— When the velocity of a moving fluid remains unchanged in magnitude and direction at every point, we have what is called a *permanent state of fluid motion.* When the velocity of a fluid is changing at each point, we have what is called a *varying state of fluid motion.* Thus, when an orifice in a large tank of water is suddenly opened, a perceptible time elapses before the jet of water becomes established. During this time the velocity of the water is changing rapidly at each point in the jet. After the jet becomes steady, however, the velocity of the water at each point remains constant in magnitude and in direction. The magnetic field in the neighborhood of a moving magnet or in the neighborhood of a moving or changing electric current is an example of a varying vector distribution.

Rate of change of a distributed vector at a point.—Let the line α, Fig. 105, represent the value at a given instant, of the velocity of a fluid at the point p, and let the line $\alpha + \Delta\alpha$ represent the

velocity of the fluid at the same point after a time-interval Δt has elapsed. The limiting value of $\frac{\Delta\alpha}{\Delta t}$ as Δt approaches zero is the rate of change of α at the point p.* This rate of change of α has a definite value and is in a definite direction at each point of space and it is therefore a distributed vector.

123. Line integral of a distributed vector.—Consider a line or path pp' in a vector field as shown in Fig. 106. Let Δs be an element of this line or path, let R be the value of the distributed

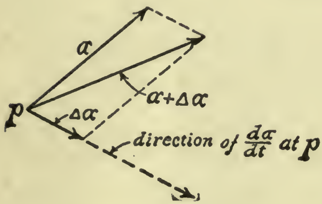


Fig. 105.

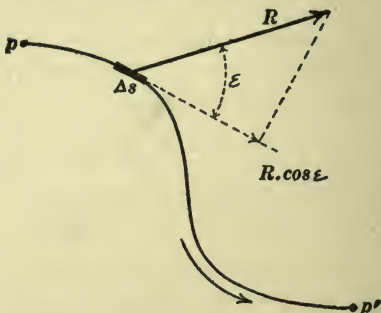


Fig. 106.

vector at the element Δs, and let ϵ be the angle between R and Δs. Then $R \cos \epsilon$ is the resolved part of R parallel to Δs, and $R \cos \epsilon \cdot \Delta s$ is the scalar part of the product of R and Δs; and:

$$E = \int R \cos \epsilon \cdot ds \tag{1}$$

is called the *line integral* of the distributed vector R along the line or path over which this summation is extended. The angle ϵ is reckoned between R and the positive direction of Δs, the positive

* In this illustration the velocity under consideration is the changing velocity of the successive particles of the fluid as they pass the point p, not the changing velocity of a given particle while it is traveling along near p. The latter velocity may be changing from instant to instant even though the former is invariable.

direction of Δs being the direction in which Δs would be passed over in traveling along the line pp' in a chosen direction. If the chosen direction be changed, $\cos \epsilon$ will change sign at each element. Therefore the line integral from p to p' is equal to the line integral from p' to p in Fig. 106, but opposite in sign.

Examples.—The line integral of electric field along a path is called the *electromotive force* along the path. The line integral of a magnetic field along a path is called the *magnetomotive force* along the path. The line integral of fluid velocity along a path is called the *circulation* of the fluid along the path.

Cartesian expression for line integral.—Let X, Y and Z be the components of the vector R, and let dx, dy and dz be the components of the line element ds. Then we have:

$$R = X + Y + Z \quad \text{(a vector equation)} \tag{2}$$

and

$$ds = dx + dy + dz \quad \text{(a vector equation)} \tag{3}$$

The product of the two vectors R and ds is part scalar and part vector as explained in Arts. 114 and 115, and the scalar part of the product is $R \cos \epsilon \cdot ds$ or $X \cdot dx + Y \cdot dy + Z \cdot dz$. Therefore equation (1) may be written:

$$E = \int (X \cdot dx + Y \cdot dy + Z \cdot dz) \tag{4}$$

Line integral of the gradient of a distributed scalar.—Consider a distributed scalar ψ. Let us call it temperature for the sake of intelligibility. Let R in Fig. 106 be the gradient of ψ, that is R is the temperature gradient. Then the line integral of R along the path pp' is equal to $\psi' - \psi$, where ψ' is the temperature at p' and ψ is the temperature at p. This is evident from the following considerations. The product $R \cos \epsilon$ in Fig. 106 is the resolved part of the temperature gradient in the direction of the line element Δs, and $R \cos \epsilon \cdot \Delta s$ is the change of temperature along Δs. Therefore $\Sigma R \cos \epsilon \cdot \Delta s$ is the sum of the changes of temperature along all parts of the path pp' or the total change of temperature from p to p'.

Line integral of a gradient along co-terminous paths.—The line integral of R along any path pp' is the temperature difference between p and p'. Therefore the line integral of R is the same for all paths from p to p'. There is an important exception to this proposition as follows:

Imagine an ordinary auger to be placed with its axis at right angles to the plane of the paper in Fig. 107, the plane of the

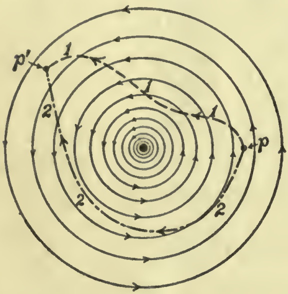


Fig. 107.

Slope-lines on an auger-hill.

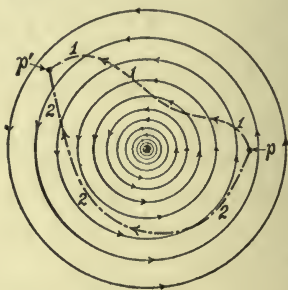


Fig. 108.

Magnetic lines of force around an electric wire.

paper being the base plane and the winding-stair-like surface of the auger being looked upon as the surface of a hill raised above the base plane. This hill we will call an auger-hill, and its slope lines are shown by the fine-line circles in Fig. 107. The height of this auger-hill above any point p has many values differing from each other by, say, one inch, where one inch is the "pitch" of the auger.

Now the line integral of the slope of a hill along any path is the difference of level of the ends of the path. A path which returns to its starting point (a *closed path*) comes back to its initial level on an ordinary hill, but a path from p back to p in Fig. 107 does not come back to its initial level if the path circles round the axis of the auger. Therefore the line integral of slope along the path **1** from p to p' in Fig. 107 is not the same as the line integral of slope along path **2** from p to p'.

The fine-line circles in Fig. 108 are the magnetic lines of force (slope lines of the magnetic-potential hill) in the neighborhood of an electric wire perpendicular to the plane of the paper. The line integral of the magnetic-potential slope along path 1 from p to p' in Fig. 108 is not the same as the line integral of the magnetic-potential slope along path 2 from p to p'.

124. Potential of a vector field.—Consider any distributed vector R. Then the "height" at each point of space of an imagined "hill" whose slope or gradient is everywhere equal to R is called the *potential* of R. Thus the temperature at each point of a body is the "height" at that point of a hill whose slope is everywhere equal to the temperature gradient, therefore temperature is the *potential* of temperature gradient. The "height" in volts at each point of space of an imagined "hill" whose slope is everywhere equal to the electric field intensity (in volts per centimeter) is called *electric potential*. The "height" at each point of space of an imagined "hill" whose slope is everywhere equal to the velocity of a moving fluid is called the *velocity potential* of the fluid.

The temperature at a point of a body can of course be determined by placing a thermometer at that point. That is, temperature is an actual physical condition. The electric potential at a point, however, is not an actual measurable physical condition at that point; indeed electric potential exists only in the imagination as a sort of mathematical fiction whose usefulness grows out of the fact that it is often very helpful to think of a given vector field as a gradient. An example showing the usefulness of the idea of potential is given in Art. 133.

Theorem as to the existence of potential.—Let ψ be a distributed scalar, like temperature. Then the component gradients of ψ are given by equation (1), (2) and (3) of Art. 119. Differentiating the first of these equations with respect to y and the second with respect to x we have:

$$\frac{d^2\psi}{dy \cdot dx} = \frac{dX}{dy} \tag{1}$$

and

$$\frac{d^2\psi}{dx \cdot dy} = \frac{dY}{dx} \tag{2}$$

But $\frac{d^2\psi}{dy \cdot dx}$ and $\frac{d^2\psi}{dx \cdot dy}$ are always identical as stated in Art. 59 and as demonstrated in Art. 134. Therefore from equations (1) and (2) we have:

$$\frac{dX}{dy} = \frac{dY}{dx}$$

or

$$\frac{dX}{dy} - \frac{dY}{dx} = 0 \tag{3}$$

Similarly from equations (2) and (3) of Art. 119, we obtain:

$$\frac{dY}{dz} - \frac{dZ}{dy} = 0 \tag{4}$$

and from equations (1) and (3), we obtain:

$$\frac{dZ}{dx} - \frac{dX}{dz} = 0 \tag{5}$$

In these equations (3), (4) and (5) X, Y and Z are the component gradients of any scalar function ψ, and *any distributed vector whose components satisfy equations* (3), (4) *and* (5) *can have a potential, or, in other words, any distributed vector whose components satisfy* (3), (4) *and* (5) *can be looked upon as the gradient of an imagined "hill."*

Multivalued potential.—The auger-hill which is represented in Fig. 107 has a multiplicity of heights above any point p. Similarly the magnetic-potential in Fig. 108 has a multiplicity of values at any point p. Of course the "height" of the potential "hill" in Fig. 108 may be thought of as an actual height measured upward from the plane of the paper; but one must not forget that the magnetic field under consideration fills all space, and that there is a multiplicity of values of magnetic potential at each point in space. See Art. 66.

125. Surface integral of a distributed vector. Flux.— Let $AOOB$, Fig. 109, be a diaphragm stretching across a closed loop of wire, the dots A and B being where the wire passes through the plane of the paper. Let ΔS be the area of an ele-

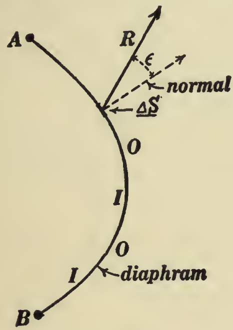


Fig. 109.

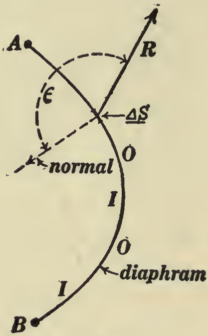


Fig. 110.

ment of the diaphragm, let R be the value at ΔS of a distributed vector, and let ϵ be the angle between R and the normal to ΔS this normal being always drawn outward from the same side OO of the diaphragm. Then $R \cos \epsilon$ is the resolved part of R normal to ΔS, and $R \cos \epsilon \cdot \Delta S$ is the scalar part of $R \cdot \Delta S$; and:

$$\Phi = \int R \cos \epsilon \cdot dS \tag{1}$$

is called the *surface integral* of R over the portion of the diaphragm over which the integration is extended.

If the normal to ΔS in Fig. 109 is reversed, as shown in Fig 110, then $\cos \epsilon$ will be reversed in sign, and the surface integral, retaining its numerical value, will be reversed in sign.

In the integration over a closed surface like a box or sphere, the normal is understood, throughout the following discussion, to be drawn outwards.

Examples.—If R in Fig. 109 is the velocity of a fluid at ΔS, then $R \cos \epsilon$ is the resolved part of the velocity normal to ΔS,

$R \cos \epsilon \cdot \Delta S$ is the volume of fluid per second flowing across ΔS, and $\int R \cos \epsilon \cdot dS$ is the total volume of fluid per second flowing through the loop of wire AB. The volume of fluid per second passing through a loop is called the *flux* of the fluid through the loop, and the surface integral of any distributed vector over a surface is called the *flux* of the vector across the surface. Thus *magnetic flux* is the surface integral of magnetic field and *electric flux* is the surface integral of *electric field.*

The surface integral of a distributed vector over a closed surface like a box or sphere is called the flux *into* or *out of* the region enclosed by the box or sphere.

Cartesian expression for surface integral.—Let X, Y and Z be the components of the vector R in Fig. 109, and let da, db and dc be the areas of the projections of dS on the yz, on the xz, and on the xy planes, respectively. Then:

$$R = X + Y + Z \quad \text{(a vector equation)} \tag{2}$$

and

$$dS = da + db + dc \quad \text{(a vector equation)} \tag{3}$$

The scalar part of the product $R \cdot dS$ is

$$X \cdot da + Y \cdot db + Z \cdot dc \; (= R \cos \epsilon \cdot dS).$$

But the shapes of the surface elements da, db and dc may be anything whatever. Therefore we may write $dy \cdot dz$ for da, $dx \cdot dz$ for db, and $dx \cdot dy$ for dc. Hence equation (1) becomes:

$$\Phi = \int\int (X \cdot dy \cdot dz + Y \cdot dx \cdot dz + Z \cdot dx \cdot dy) \tag{4}$$

126. Divergence of a distributed vector.—In some cases a distributed vector "flows" outwards or *emanates* from a region of space. Thus the liquid in a tank flows outwards from the end of a supply pipe. When a gas is expanding each portion of the gas is growing less dense and there is an outward flow from every small part of the region occupied by the expanding gas. What is called electric field emanates from electrically charged bodies,

and when electric charge is spread throughout a region, electric field emanates from every small part of the region occupied by the charge.

Consider a small volume element $\Delta\tau$ in the neighborhood of a point p in a vector field. Let $\Delta\Phi$ be the flux of the distributed vector R *out of* the volume element $\Delta\tau$. It can be shown,* when R is a continuous function of the coördinates x, y and z, that the ratio $\frac{\Delta\Phi}{\Delta\tau}$ approaches a definite limiting value as the volume element $\Delta\tau$ grows smaller and smaller. This limiting value, $\frac{d\Phi}{d\tau}$ is called the *divergence* of the vector field at the point p. Therefore, representing the divergence of a vector field at a point by ρ, we have:

$$d\Phi = \rho \cdot d\tau \tag{1}$$

in which $d\Phi$ is the flux of R coming out of the volume element $d\tau$.

When a vector field flows *into* each small part of a region, the divergence is negative. Negative divergence is sometimes called *convergence.*

The flux Φ of a vector field across a surface is a scalar quantity, as is evident from the discussion of equation (1) of Art. 125. Also volume is a scalar quantity. Therefore divergence $\left(\frac{\Delta\Phi}{\Delta\tau}\right)$ is a scalar quantity; and since a vector field has a definite divergence at each point (of course the divergence may be zero) of space, it is evident that the divergence of a vector function (a distributed vector) is a scalar function (a distributed scalar).

Cartesian expression for divergence.—Consider a small cube of which the edges are dx, dy and dz as shown in Fig. 111. Let X, Y and Z be the components of the given distributed vector R

* The actual proof that $\frac{\Delta\Phi}{\Delta\tau}$ has a definite limiting value may be established without great difficulty by considering the portion v'' of the linear vector field in Art. 133.

at the point p. Then the flux of R *into* the cube across the face a is $X \cdot dy \cdot dz$, because X is the component of R normal to the face a and $dy \cdot dz$ is the area of face a. If X is the x-component of R at any point of the face a, then $\left(X + \frac{dX}{dx} \cdot dx\right)$

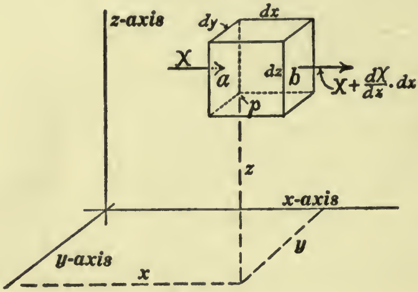


Fig. 111.

is the x-component of R at the corresponding point of the face b, and if $X \cdot dy \cdot dz$ is the flux *into* the cube across face a then $\left(X + \frac{dX}{dx} \cdot dx\right) \cdot dy \cdot dz$ is the flux *out of* the cube across face b.* Therefore the flux of R *out of* the cube across face b exceeds the flux of R *into* the cube across face a by the amount $\frac{dX}{dx} \cdot dx \cdot dy \cdot dz$. In the same manner it may be shown that the net flux of R *out of* the cube across the two faces which are perpendicular to the y-axis is $\frac{dY}{dy} \cdot dx \cdot dy \cdot dz$; and that the net flux *out of* the cube across the two faces which are perpendicular to the z-axis is $\frac{dZ}{dz} \cdot dx \cdot dy \cdot dz$. Therefore the total flux out of the cube is:

$$d\Phi = \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) \cdot \ dx \cdot dy \cdot dz \tag{2}$$

* The argument here given is not entirely rigorous, and the peculiar wording of this sentence is intended to suggest a form of argument which is rigorous.

But $dx \cdot dy \cdot dz$ is the volume $d\tau$ of the cube. Therefore, using equation (1) we have:

$$\text{div} = \rho = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} \qquad (3)$$

PROBLEMS.

1. All space is filled with a fluid moving parallel to the x-axis of reference at a uniform velocity of 10 centimeters per second. Find an expression for the velocity potential of the fluid. Ans. $\psi = 10x +$ any constant.

2. The velocity components of a moving fluid parallel to the axes of reference are everywhere equal to a, b and c respectively. Find the velocity potential. Ans. $\psi = ax + by + cz +$ any constant.

3. The velocity components of a fluid parallel to the axes of reference are ax, by and cz, respectively. Find the velocity potential. Ans. $\psi = \frac{1}{2}ax^2 + \frac{1}{2}by^2 + \frac{1}{2}cz^2 +$ any constant.

4. A viscous fluid flowing over a plane has a velocity which is everywhere given by the equation $X = ay$ as shown in Fig. $p4$. Show that no velocity potential exists.

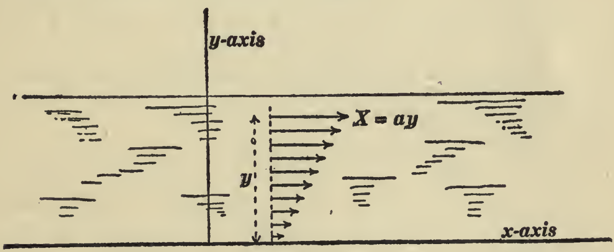


Fig. $p4$.

5. A vessel of water rotates uniformly at a speed of two revolutions per second about a vertical axis (the z-axis). Derive expressions for the velocity components of the moving water and show that the velocity has no potential.

6. Liquid flows over an infinite plane towards a circular spot

20 centimeters in radius where the liquid leaks through the plane at the rate of two cubic centimeters per second for each square centimeter of the leaky portion of the plane. The depth of the liquid is everywhere 10 centimeters. Choose the x and y-axes of reference in the plane with the origin at the center of the leaky portion, and derive expressions for the x and y-components of the fluid velocity in the region over the leaky portion. Ans. x-component $= \frac{x}{10}$.

Note.—In this and the following problems consider only the horizontal part of the motion and assume the velocity to be the same from top to bottom of the liquid.

7. Find an expression for the velocity potential in the region over the leaky portion. Ans. $\frac{x^2 + y^2}{20}$ + any constant.

8. Derive expressions for the x and y components of the fluid velocity in the region outside of the leaky portion. Ans. x-component $= \frac{40x}{x^2 + y^2}$.

9. Find an expression for the velocity potential in the region outside of the leaky portion. Ans. $20 \log_e (x^2 + y^2)$ + any constant.

10. Determine the constants of integration involved in the answers to problems 7 and 9 on the assumption that the velocity potential is zero at the edge of the leaky spot. Ans. (*a*) -20, (*b*) $-20 \log_e 400$.

11. A straight "fence" 20 centimeters long and 10 centimeters high is placed in the layer of moving liquid on the plane in problem 6. Find the surface integral of the fluid velocity over the fence (*a*) when the ends of the fence are at a distance of $\sqrt{125}$ centimeters from the center of the leaky spot, and (*b*) when the ends of the fence are $\sqrt{1{,}000}$ centimeters from the center of the leaky spot. Ans. (*a*) 100 cubic centimeters per second, (*b*) 800 $\tan^{-1} 0.333$ cubic centimeters per second.

12. Find the divergence of the fluid velocity in problem 6 (*a*) over the leaky spot and (*b*) in the region outside of the leaky spot.

Ans. (a) 0.2 cubic centimeter per second per cubic centimeter, (b) zero.

13. Find the divergence of the fluid velocity specified in problem 3. Ans. $a + b + c$.

14. One end of a rubber band is fixed, and the band is stretched by moving the other end of the band at a velocity of 2 centimeters per second. The band is 24 centimeters long at a given instant. Find the divergence of v where v is the distributed velocity of the band. Assume that the band does not contract laterally. Ans. 0.0833 cubic centimeters per second per cubic centimeter.

15. Find the distribution of pressure in a tank of water rotating at a speed of one revolution per second ($\omega = 2\pi$ radians per second), the density of water being 62.5 pounds per cubic foot. Ans. $p = 123.3\ r^2$ where p is the pressure in poundals per square foot at a point r feet from the axis of rotation.

Note.—Consider an element of the rotating liquid as indicated by the shaded area in Fig. p15, the dimension perpendicular to the plane of the paper being l. The area of side a is $lr\ .\ d\theta$ and the outward push on a is $plr\ .\ d\theta$. The area of the side b is $l(r + dr)\ .\ d\theta$, and the pressure at this face is $(p + dp)$; therefore, dropping infinitesimals of the third order, the force pushing inwards on face b is $plr\ .\ d\theta + pl \cdot dr\ .\ d\theta + lr\ .\ pd\ .\ d\theta$. The area of faces c and e is $l\ .\ dr$ and the pressure over these faces may be considered as equal to p; therefore the normal forces on faces c and e are $pl\ .\ dr$ as shown. The outward component of these two forces, according to

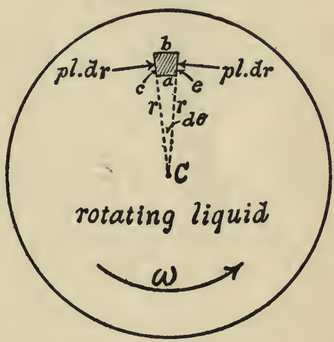


Fig. p15.

Art. 51, is $\dfrac{pl\ .\ dr}{r} \times r\ .\ d\theta$ (omitting infinitesimals of the third order). Therefore the net inward force due to pressure acting on the element of liquid is $lr\ .\ dp\ .\ d\theta$, and this must be equal to the product of the mass $Dlr\ .\ d\theta\ .\ dr$ of the element of liquid and its radial acceleration $\omega^2 r$.

16. Find the expressions for the pressure gradients in the

rotating liquid of problem 15 in the directions of x and y axes lying in a plane at right angles to the axis of rotation. Ans. $X = 246.6x$, $Y = 246.6y$.

17. Find the divergence of the pressure gradient in the rotating fluid of problem 15. Ans. 493.2.

127. Reduction of a surface integral over a closed surface to a volume integral extended throughout the enclosed region.— A very important transformation in the theory of electricity and magnetism is the reduction of a surface integral to a volume integral or *vice versa*. Let R be a distributed vector, and let us consider its surface integral over a closed surface (normal directed outwards). Imagine the entire enclosed region to be broken up into small cells like the individual bubbles in a mass of foam. *Then the integral of R over the bounding surface of the region is equal to the sum of the integrals of R over the bounding surfaces of the individual cells* (normal directed outwards in each case). This proposition is evident when we consider that every wall which separates two cells is integrated over twice with directions of normal opposite (see Art. 125), so that the surface integrals over dividing walls are thus cancelled. The only surface integrals which are not thus cancelled are the integrals over the parts of the surface which bounds the region.

Let $d\Phi$ be the surface integral over one of the cells (the flux of R out of the cell), and let $\int R \cos \epsilon \cdot dS$ be the surface integral of R over the bounding surface of the enclosed region. Then from the above proposition we have:

$$\int R \cos \epsilon \cdot dS = \int d\Phi \tag{1}$$

But $d\Phi = \rho \cdot d\tau$ according to equation (1) of Art. 126, so that equation (1) becomes:

$$\int R \cos \epsilon \cdot dS = \int \rho \cdot d\tau \tag{2}$$

This equation may be expressed in words as follows: *The integral of a distributed vector R over a closed surface is equal to the integral*

of the divergence of R *throughout the enclosed region.* The first member of (2) is of course a surface integral and the second member is a volume integral.

128. Solenoidal vector fields.—A vector field is said to be *solenoidal* when its divergence is everywhere zero. The surface integral of such a vector field over any closed surface is zero according to equation (2) of Art. 127.

Tube of flow.—Imagine stream lines to be drawn in a vector field from each point of the periphery of a closed curve or loop. These stream lines form a tubular surface which is called a *tube of flow* of the given distributed vector. Consider a number of diaphragms which stretch across a tube of flow. The flux is the same across each diaphragm. This is evident when we consider that any two diaphragms, together with the walls of the tube, constitute a closed surface out of which the total flux must be zero because the distributed vector under consideration is assumed to be solenoidal; but the flux across the walls of the tube is zero because the vector field is everywhere parallel to the walls. Therefore the flux into the enclosed space across one diaphragm must be equal to the flux out of the enclosed space across the other diaphragm.

Unit tube.—A tube of flow is called a *unit tube* when the flux through the tube is unity. Thus, for example, a unit tube has a sectional area of 0.1 of a square inch at a point in a moving fluid where the velocity of the fluid is 10 inches per second. A unit tube has a sectional area of 0.1 of a square centimeter at a point in a magnetic field where the intensity of the field is 10 gausses. Imagine the solenoidal region of a distributed vector to be divided up into unit tubes. Then the flux across any surface anywhere in the region will be equal to the number of these unit tubes which pass through the surface. Each unit tube may be conveniently represented in imagination by the single stream line along the axis of the tube. When stream lines are drawn in this way, so that each stream line represents a unit tube, then the flux across any

surface in the vector field is equal to the number of stream lines which pass through the surface. The lines of force in a magnetic or electric field are always thought of as being drawn so that each line represents a unit tube, and the quantity of magnetic flux or electric flux through a surface is expressed by the number of lines of force which pass through the surface.

129. Curl of a distributed vector.—In Art. 126 examples were given of vector fields which emanate or "flow out" from a region or from every small part of a region. There are important cases in which a vector field curls round a region or round every small part of a region. Thus the stream lines in a rotating bowl of water curl round the central portions of the bowl. The magnetic field due to an electric wire curls round the wire. The electric field induced by an iron rod while the rod is being magnetized curls round the rod.

Consider a small plane area ΔS at a point p in a vector field. Let ΔL be the line integral of the distributed vector R round the boundary of this element of area. It can be shown,* when R is a continuous function of the coördinates x, y and z, that the ratio $\frac{\Delta L}{\Delta S}$ approaches a definite limiting value as the element of area ΔS grows smaller and smaller. This limiting value, $\frac{dL}{dS}$ is the component of a new vector† C in the direction of the normal to dS, and this new vector C is called the *curl* of the given vector R at the point p. From this definition we have:

$$dL = C \cos \epsilon \cdot dS \tag{1}$$

where dL is the line integral of R round the boundary of a

* The actual proof that $\frac{\Delta L}{\Delta S}$ approaches a definite limiting value may be established without great difficulty by considering the part v' of the linear vector field in Art. 133; see Art. 135.

† See Art. 135.

small plane element of area dS, C is the curl of R at the element of area, and ϵ is the angle between C and the normal to dS.

Example of curl.—Consider a uniformly rotating bowl of water, the speed of rotation being ω radians per second. Consider the character of the fluid motion in the immediate neighborhood of the point p distant D from the axis of rotation of the bowl as shown in Fig. 112. The motion of a small portion of the water near p may be thought of as a combination of (1) A motion of translation at velocity $D\omega$, and (b) A simple motion of rotation at a speed of ω radians per second about an axis through the point p.* Now in considering the line integral of the fluid velocity around the circle cc, the motion of translation evidently need not be considered. Let r be the radius of the circle cc. The velocity of the fluid at every point of the circumference of cc (rotatory motion only being considered) is ωr and it is everywhere in the direction of the circumference. Therefore the angle ϵ is zero and the expression for the line integral around the circle reduces to *circumference* $\times\ \omega r$. Therefore $\Delta L = 2\pi r^2\omega$, where ΔL is the line integral of the fluid velocity around the circle cc. But the area of the circle is $dS = \pi r^2$. Therefore from equation (1) we get: $C = 2\omega$. That is, the curl of the fluid velocity in the rotating bowl is equal to two times the speed of the bowl in radians per second; and since the expression for C does not contain D it is evident that C has the same value everywhere in the bowl.

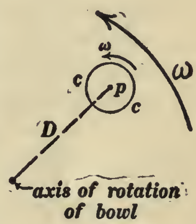


Fig. 112.

Now the angular velocity ω of the bowl is a vector and its vector direction is parallel to the axis of rotation and towards the reader in Fig. 112. Therefore C is a vector, and it is towards the reader at every point in Fig. 112. The "stream lines" of the vector C are straight lines parallel to the axis of rotation of the bowl.

* See Franklin and MacNutt's *Mechanics and Heat*, pages 174–175, The Macmillan Co., New York, 1910.

Cartesian expression for curl.—Consider a small rectangular plane area of which the edges are dy and dz, as shown in Figs. 113 and 114. Let X, Y and Z be the components at p of the

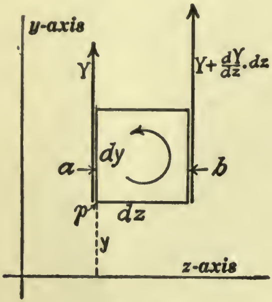


Fig. 113.

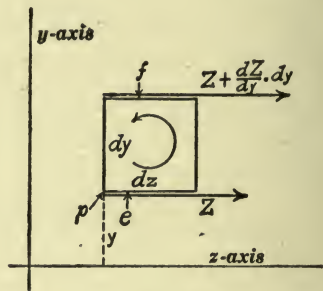


Fig. 114.

given distributed vector R. To find the line integral of R around the small rectangle in the direction of the small curved arrow, let us consider first the two edges a and b in Fig. 113. The line integral along the edge a is $-Y \cdot dy$ because the component Y is counter to the direction of the small curved arrow. If Y is the y-component of R at any point of the edge a, then the y-component of R at the corresponding point of the edge b being $\frac{dY}{dz} \cdot dz$ greater, is equal to $Y + \frac{dY}{dz} \cdot dz$; and if the line integral of R along edge a is $-Y \cdot dy$, then the line integral of R along the edge b is $+\left(Y + \frac{dY}{dz} \cdot dz\right) \cdot dy$.* Therefore the net value of the line integral along the two edges a and b is $\frac{dY}{dz} \cdot dy \cdot dz$. In the same way it may be shown from Fig. 114 that the net line integral (in the direction of the small curved arrow) along the edges e and f is $-\frac{dZ}{dy} \cdot dy \cdot dz$. Therefore

* The argument here given is not entirely rigorous, and the peculiar wording of this sentence is intended to suggest a slightly modified argument which is rigorous.

the total line integral dL of R around the small rectangle is:

$$dL = \left(\frac{dY}{dz} - \frac{dZ}{dy}\right) \cdot dy \cdot dz \tag{2}$$

But $dy \cdot dz$ is the area dS of the rectangle. Therefore using equation (1) and remembering that the x-component of the curl is at right angles to the rectangle in Figs. 113 and 114, we have:

$$C_x = \frac{dY}{dz} - \frac{dZ}{dy} \tag{3}$$

In a similar manner we may get:

$$C_y = \frac{dZ}{dx} - \frac{dX}{dz} \tag{4}$$

and

$$C_z = \frac{dX}{dy} - \frac{dY}{dx} \tag{5}$$

where C_x, C_y and C_z are respectively the x, y and z components of the curl of the distributed vector R whose x, y and z components are X, Y and Z respectively.

The curl of a gradient is necessarily equal to zero.— This is evident when we consider that equations (3), (4) and (5) of Art. 124 are satisfied in a vector field when the vector is a gradient.

130. Reduction of a line integral around a closed loop to a surface integral over a diaphragm bounded by the loop.—A very important transformation in the theory of electricity and magnetism is the transformation of a line integral to a surface integral or *vice versa*. Let R be a distributed vector, and let us consider its line integral around a closed curve or loop AB, Fig. 115, the heavy arrow showing the direction in which the line integral is taken. Imagine a diaphragm of any shape whatever stretched across the loop AB, and imagine this diaphragm to be divided up into small meshes as if the diaphragm were made of wire

gauze. *Then the integral of* R *round the loop* AB *is equal to the sum of the integrals of* R *round the individual meshes of the diaphragm*, the integrals round the meshes being taken in the same direction as the integral round the loop as indicated by the small curled arrows. This proposition is evident when we consider that every line dividing two meshes is integrated over twice in opposite directions (see Art. 123), and when we consider that in integrating round the various meshes we eventually integrate along every portion of the loop AB once in the direction of the heavy arrow in Fig. 115.

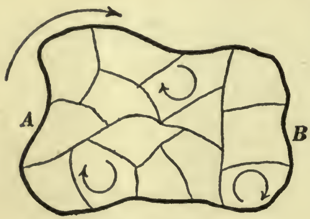


Fig. 115.

Let dL be the line integral of R round one of the meshes, and let $\int R \cos \epsilon \cdot ds$ be the line integral of R round the loop AB. Then from the above proposition we have:

$$\int R \cos \epsilon \cdot ds = \int dL \tag{1}$$

But $dL = C \cos \epsilon \cdot dS$, according to equation (1) of Art. 129, so that equation (1) becomes:

$$\int R \cos \epsilon \cdot ds = \int C \cos \epsilon \cdot dS \tag{2}$$

This equation may be expressed in words as follows: *The integral of a distributed vector* R *round a closed loop is equal to the integral of the curl of* R *over any diaphragm stretched across the loop.* The first member of (2) is of course a line integral and the second member is a surface integral.

The divergence of the curl of a distributed vector is always and everywhere equal to zero.—Consider two diaphragms stretched across a closed loop. The integral of C is the same over both of these diaphragms because each surface integral is equal to the line integral of R round the loop. If the direction of the normal

be reversed in the integral over one of the diaphragms the integral will be reversed in sign (see Art. 125). Therefore the surface integral of C over a closed surface, namely, the two diaphragms, with normal directed outwards, is equal to zero. That is C is a vector whose flux *out of* or *into* a closed surface is always equal to zero. Therefore C is a solenoidal vector and its divergence is everywhere zero. See Art. 127.

131. Vector potential.—Let us imagine a distributed vector of which a given distributed vector is the curl. Then the imagined distributed vector is called the *vector potential* of the given distributed vector. The divergence of a curl is always and everywhere necessarily equal to zero, as explained in the previous article. Therefore, to have a vector potential, a given distributed vector must have no divergence. But $\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}$ is the divergence of a distributed vector whose components at a point are X, Y and Z, according to equation (3) of Art. 126. Therefore the existence of a vector potential (of a given distributed vector) is determined by the condition:

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0 \tag{1}$$

where X, Y and Z are the components at a point of the given distributed vector.

Example.—In certain cases of fluid motion each particle of the fluid is rotating at a definite speed about a definite axis. This spinning motion of the particles of a fluid is a distributed vector* and it is called *vortex motion.* Now it was shown in Art. 129, that the curl of the velocity of the water at a point p in a rotating bowl is equal to two times the spin velocity ω of the particle of water at p. That is, ignoring the factor 2, *the curl of fluid velocity*

* Spin is a vector quantity, its vector direction being along the axis of spin in the direction in which a right-handed screw would travel if turned in the direction of the spin.

is the vortex motion; therefore the vortex motion of a fluid is a distributed vector whose vector potential is the fluid velocity. In this case the vector potential (of vortex motion) is physically existent; in general, however, the vector potential of a given distributed vector is merely imagined to exist. The idea of vector potential is useful because it is sometimes helpful to think of a given distributed vector as a curl.

132. Rotational and irrotational vector fields. Vector potential and scalar potential.—When the velocity of a moving fluid has curl the particles of the fluid are in rotation. In consequence of this fact any vector field which has curl is called a *rotational vector field.* A vector field which has no curl is called an *irrotational vector field.*

In an irrotational vector field equations (3), (4) and (5) of Art. 129 reduce to equations (3), (4) and (5) of Art. 124. Therefore an irrotational vector field can be thought of as a gradient, or, in other words, an irrotational vector field has a potential (a scalar potential).

In a rotational vector field equations (3), (4) and (5) of Art. 124 are not satisfied, and therefore a rotational vector field cannot be thought of as a gradient, or, in other words, a rotational vector field has no potential (no scalar potential).

A vector field which has no divergence is called a *solenoidal vector field,* and a solenoidal vector field can be thought of as a curl, or, in other words, a solenoidal vector field has a vector potential.

A vector field which has divergence is called a non-solenoidal vector field, and such a vector field cannot be thought of as a curl, or, in other words, a non-solenoidal vector field has no vector potential.

PROBLEMS.

1. A viscous liquid moves over the xy plane so that $X = az$. Find the curl. Ans: $-a$.

2. Find the vector potential of fluid velocity when the fluid is everywhere moving in the direction of the z-axis at a velocity

of $+ 100$ centimeters per second. Ans. $100z$ centimeters squared per second squared.

Note.—A distributed vector which has neither divergence nor curl in a given region may have a scalar potential and it may have a vector potential.

133. Character of a very small portion of any vector field.—Let X, Y and Z be the components of any distributed vector v at a point p whose coördinates are x, y and z. Then X, Y and Z are functions of x, y and z, and indeed we will assume them to be continuous functions. If one is to reach a clear understanding of divergence and curl it is necessary to examine carefully into the manner of distribution of v in a very small region near a chosen point. It is most convenient to locate the origin of coördinates at the chosen point. Then the coördinates x, y and z of any point in the very small region are infinitesimals, and the squares and products and higher powers of x, y and z are negligible.

Now the components X, Y and Z being continuous functions of x, y and z can be expanded by Maclaurin's theorem as explained in Art. 90, and all terms in these expansions which contain squares and products and higher powers of x, y and z may be discarded, giving:

$$X = X_0 + a_1x + a_2y + a_3z \tag{1}$$

$$Y = Y_0 + b_1x + b_2y + b_3z \tag{2}$$

and

$$Z = Z_0 + c_1x + c_2y + c_3z \tag{3}$$

in which the constant coefficients are the values of the derivatives at the origin, as shown in the following schedule:

$$\left.\begin{array}{lll} a_1 = \dfrac{dX}{dx} & a_2 = \dfrac{dX}{dy} & a_3 = \dfrac{dX}{dz} \\[2ex] b_1 = \dfrac{dY}{dx} & b_2 = \dfrac{dY}{dy} & b_3 = \dfrac{dY}{dz} \\[2ex] c_1 = \dfrac{dZ}{dx} & c_2 = \dfrac{dZ}{dy} & c_3 = \dfrac{dZ}{dz} \end{array}\right\} \tag{I}$$

The "linear" vector field.—The simplest type of vector field is the homogeneous field; in such a field X has the same value throughout the field, Y has the same value throughout the field, and Z has the same value throughout the field. When X, Y and Z are linear functions of x, y and z we have what is called a *linear* vector field. Thus equations (1), (2) and (3) represent a linear vector field.

Resolution of a linear vector field into simple parts.—The discussion of divergence and curl in Arts. 126 and 129 is not rigorous, and, as is always the case in a merely plausible discussion, the harder one tries to understand it the more vague and unintelligible it becomes. It is now proposed, therefore, to establish rigorously the ideas of divergence and curl by considering a linear vector field as expressed by equations (1), (2) and (3). For this purpose it is necessary to resolve the given linear vector field into three parts, and to be able to speak of these parts intelligibly let us think of the given linear vector field as the velocity v of a moving fluid, the components of v being X, Y and Z as given in equations (1), (2) and (3). The three parts into which v is to be resolved are:

1. A uniform translatory motion of the entire body of fluid. This part of v will be represented by v_0 and its components are X_0, Y_0 and Z_0.

2. A simple motion of rotation of the entire body of fluid. This part of v will be represented by v' and its components will be represented by X', Y' and Z'.

3. A continuous stretching of the entire body of fluid in three mutually perpendicular directions. This part of v will be represented by v'' and its components will be represented by X'', Y'' and Z''.

Discussion of v′.—Consider a rotating body of which the axis of rotation passes through the origin of coördinates, and let ω_x, ω_y and ω_z be the components of the angular velocity of the body around the x, y and z axes of reference, respectively.

Consider a point p of the body of which the coördinates are x, y and z as indicated in Figs. 116, 117 and 118. The velocity

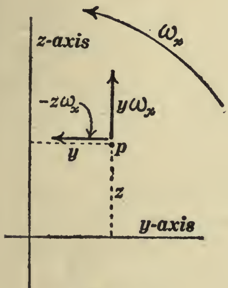


Fig. 116.

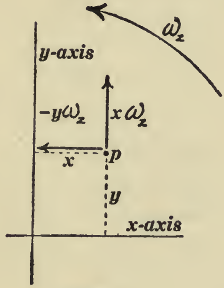


Fig. 117.

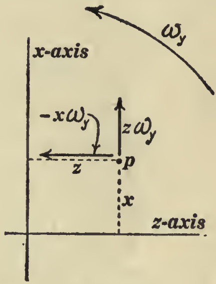


Fig. 118.

of p due to ω_x consists of two components $y\omega_x$ and $-z\omega_x$ as indicated in Fig. 116, and similar statements may be made concerning Figs. 117 and 118. Therefore, picking out the x-components of the velocity of p in Figs. 117 and 118, we get the value of X', and in a similar manner we get expressions for Y' and Z' as follows:

$$X' = \cdots - \omega_z y + \omega_y z \tag{4}$$
$$Y' = + \omega_z x \cdots - \omega_x z \tag{5}$$
$$Z' = - \omega_y x + \omega_x y \cdots \tag{6}$$

The values of ω_x, ω_y and ω_z, expressed in terms of the coefficients in equations (1), (2) and (3), can best be determined after the expressions for v'' have been formulated.

Discussion of v''.—As in case of v' it is necessary to find the *forms* of expressions which give X'', Y'' and Z''. For this purpose consider three mutually perpendicular axes of reference x_1, y_1 and z_1. Then a continuous stretch of the portion of fluid parallel to the x_1-axis is represented by the equation $X_1 = \alpha x_1$, where X_1 is a velocity parallel to the x_1-axis and α is a constant. Analogous expressions using β and γ as constants represent continuous stretches parallel to the y_1 and z_1 axes. Therefore

continuous stretches in three mutually perpendicular directions (which directions are for the moment chosen as the axes of reference) are expressed by the equations:

$$X_1 = \alpha x_1 \tag{7}$$

$$Y_1 = \beta y_1 \tag{8}$$

and

$$Z_1 = \gamma z_1 \tag{9}$$

It is desired to transform these equations so as to express exactly the same fluid motion, but to express it by giving the components of v'' parallel to new axes x, y and z, and as functions of the new coördinates x, y and z. This transformation is enormously simplified by expressing the given fluid motion [equations (7), (8) and (9)] in terms of its velocity potential P, which is a distributed scalar whose x_1-gradient is $\frac{dP}{dx_1} = \alpha x_1$, whose y_1-gradient is $\frac{dP}{dy_1} = \beta y_1$ and whose z_1-gradient is $\frac{dP}{dz_1} = \gamma z_1$. Integrating these three differential equations and ignoring the constant* of integration we get:

$$P = \tfrac{1}{2}\alpha x_1^2 + \tfrac{1}{2}\beta y_1^2 + \tfrac{1}{2}\gamma z_1^2 \tag{10}$$

Now to transform equations (7), (8) and (9) as stated above we need only to substitute in equation (10) the following values for x_1, y_1 and z_1 [see equations (11), (12) and (13)], and then find the gradients of P in the directions of the new axes. In this way we get equations (14), (15) and (16).

$$x_1 = l_1 x + m_1 y + n_1 z \tag{11}$$

$$y_1 = l_2 x + m_2 y + n_2 z \tag{12}$$

$$z_1 = l_3 x + m_3 y + n_3 z \tag{13}$$

where $l_1\, m_1\, n_1$, $l_2\, m_2\, n_2$ and $l_3\, m_3\, n_3$ are the direction cosines of

* The three equations are partial differential equations, but the functions of integration reduce to a single undetermined constant.

the new x, y and z axes referred to the old x_1, y_1 and z_1 axes.

$$X'' = fx + py + qz \tag{14}$$
$$Y'' = px + gy + rz \tag{15}$$
$$Z'' = qx + ry + hz \tag{16}$$

The coefficients in these equations are of course expressions involving α, β and γ and the direction cosines in equations (11), (12) and (13), but they are represented by the single letters f, g, h, p, q and r for the sake of clearness. Anyway, the only important feature about equations (14), (15), and (16) is the *symmetry of the coefficien s* which means that X'', Y'' and Z'' are the components of a fluid motion consisting of three continuous mutually perpendicular stretches, because this degree of symmetry is the only necessary consequence of equations (7), (8) and (9).

Determination of coefficients in equations (4), (5) and (6) and in equations (14), (15) and (16) in terms of the coefficients in equations (1), (2) and (3).—Equations (4), (5) and (6) and (14), (15) and (16) show only the degree of symmetry that the coefficients must have in order that equations (4), (5) and (6) may represent a simple rotation and in order that equations (14), (15) and (16) may represent three continuous mutually perpendicular stretches, and the coefficients in equations (4), (5) and (6) and (14), (15) and (16) may be thought of as undetermined. It remains to determine these coefficients $\omega_x, \omega_y, \omega_z, f, g, h, p, q$ and r, so that v_0, v' and v'' may be component parts of the given fluid velocity v which is represented by equations (1), (2) and (3). To do this add equations (4) and (14) and place the coefficients of x, y and z in the resulting equations equal to the coefficients of x, y and z, respectively, in equation (1); and proceed in a similar manner with the other pairs of equations. In this way we get the following schedule of equations:

$$\left.\begin{array}{rrr} f = a_1 & p - \omega_z = a_2 & q + \omega_y = a_3 \\ p + \omega_z = b_1 & g = b_2 & r - \omega_x = b_3 \\ q - \omega_y = c_1 & r + \omega_x = c_2 & h = c_3 \end{array}\right\} \tag{II}$$

from which we get the following values:

$$\omega_x = \tfrac{1}{2}(c_2 - b_3)$$
$$\omega_y = \tfrac{1}{2}(a_3 - c_1)$$
$$\omega_z = \tfrac{1}{2}(b_1 - a_2)$$

Or, using the values of $a_1, a_2, a_3, b_1, b_2,$ etc., from the schedule I of equations, we have:

$$2\omega_x = \frac{dZ}{dy} - \frac{dY}{dz} \tag{17}$$

$$2\omega_y = \frac{dX}{dz} - \frac{dZ}{dx} \tag{18}$$

$$2\omega_z = \frac{dY}{dx} - \frac{dX}{dy} \tag{19}$$

and from equations II we get also:

$$f = \frac{dX}{dx} \tag{20}$$

$$g = \frac{dY}{dy} \tag{21}$$

$$h = \frac{dZ}{dz} \tag{22}$$

$$p = \tfrac{1}{2}\left(\frac{dY}{dx} + \frac{dX}{dy}\right) \tag{23}$$

$$q = \tfrac{1}{2}\left(\frac{dX}{dz} + \frac{dZ}{dx}\right) \tag{24}$$

$$r = \tfrac{1}{2}\left(\frac{dZ}{dy} + \frac{dY}{dz}\right) \tag{25}$$

PROBLEMS.

1. Find the flux of v'' out of a rectangular parallelopiped which is l feet long in the direction of the x_1-axis, w feet wide in the direction of the y_1-axis, and t feet thick in the direction of the z_1-axis, the constants in equations (7), (8) and (9) being expressed in reciprocal seconds. Ans. $lwt(\alpha + \beta + \gamma)$ cubic feet per second.

2. Find the divergence of v''. Ans. $\alpha + \beta + \gamma$.

Note.—It is here intended that the student find the flux Φ which comes out of a given volume τ, and then determine the limit of $\frac{\Phi}{\tau}$ as τ approaches zero. It is simplest to take for τ the rectangular parallelopiped of problem 1.

3. Show that the integral of v'' around any closed curve whatever is zero.

Note.—Choose the axes of reference as in equations (7), (8) and (9). Let ds be an element of the closed path or curve. Then the scalar part of the product $v'' . ds$ can easily be found, and it is easy to show that the integral of this scalar product around the closed curve is zero.

4. It is evident that the velocity v', which is a simple motion of rotation, has zero flux into or out of any closed region whatever. Therefore the divergence of v' is everywhere equal to zero. Find the line integral of v' around the ellipse of Fig. 122, the radius of the cylinder being r, the axis of the cylinder being parallel to the axis of rotation of the fluid, and the angular velocity of rotation being ω. Ans. $2\pi r^2\omega$.

134. Proof that $\frac{d^2z}{dy \cdot dx}$ and $\frac{d^2z}{dx \cdot dy}$ are identical when z is a continuous function of x and y.—This proposition may be stated so as to appeal to one's geometric sense as follows: Let z be a function of x and y and let this function be represented by a hill built upon the xy plane. Then z is the height of this hill above the point p' in the base plane, x and y being the coördinates of p' as shown in Figs. 41a and 41b in Art. 62. Let $X\left(=\frac{dz}{dx}\right)$ be the x-component of the slope of the hill at the point p (see Figs. 41a and 41b), and let $Y\left(=\frac{dz}{dy}\right)$ be the y-component of the slope of the hill at p.

Figs. 119 and 120 represent a top view of the hill. Let dz be the difference of level of the points p and q on the hill in Figs.

119 and 120. Let us travel from p to q along the sides a and b

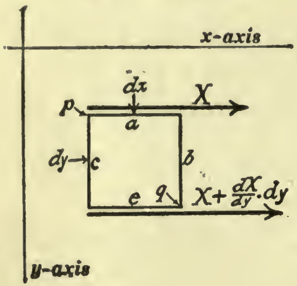


Fig. 119.

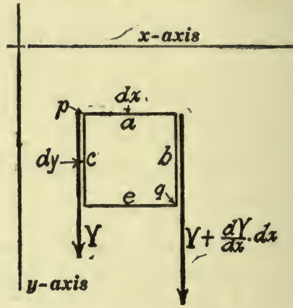


Fig. 120.

of the infinitesimal square, and the rise will be:

$$dz = X \cdot dx + \left(Y + \frac{dY}{dx} \cdot dx\right) \cdot dy$$

Let us travel from p to q along the sides c and e of the infinitesimal square, and the rise will be:

$$dz = Y \cdot dy + \left(X + \frac{dX}{dy} \cdot dy\right) \cdot dx$$

Now if z is a continuous function of x and y these two expressions for dz must be identical, that is, one must rise by the same amount in going along a and b as in going along c and e in Figs. 119 and 120. Therefore $\frac{dY}{dx}$ must be equal to $\frac{dX}{dy}$. But $X = \frac{dz}{dx}$ and $Y = \frac{dz}{dy}$. Therefore $\frac{d^2z}{dy \cdot dx}$ must be equal to $\frac{d^2z}{dx \cdot dy}$.

The above argument is not entirely rigorous but it brings out very clearly the geometrical significance of equations (3), (4) and (5) of Art. 124. To make the argument rigorous a slightly altered point of view suffices, as follows: The gradient of the hill at any point in the side e of the infinitesimal square exceeds the gradient at the corresponding point in the side a by the amount $\frac{dX}{dy} \cdot dy$, so that the rise along e exceeds the rise along a by the amount

$\left(\frac{dX}{dy} \cdot dy\right) \cdot dx$. Similarly the rise along side b exceeds the rise along side c by the amount $\left(\frac{dY}{dx} \cdot dx\right) \cdot dy$. Therefore the rise along sides c and e exceeds the rise along a and b by the amount $\left(\frac{dX}{dy} - \frac{dY}{dx}\right) \cdot dx \cdot dy$. But the rise along c and d must be the same as the rise along a and b. Therefore $\frac{dX}{dy} - \frac{dY}{dx}$ must be equal to zero.

135. To show that $\frac{\mathrm{dL}}{\mathrm{dS}}$ is the resolved part of a vector (the curl of R) in a direction normal to dS, where dL is the line integral of R around the boundary of a surface element dS.—It can be shown without difficulty that the two parts v_0 and v'' of the linear vector field v of Art. 133 have zero line integrals around any closed curve. Furthermore any distributed vector R may be looked upon as having a linear distribution throughout a very small region. Therefore to establish the above proposition it is sufficient to consider the portion v' of a linear vector field.

Now v' is a simple motion of rotation about a definite axis. Let ω be the angular velocity of this rotation. The line integral of v' around the circle cc in Fig. 121 is equal to $2\pi r^2\omega$, so that

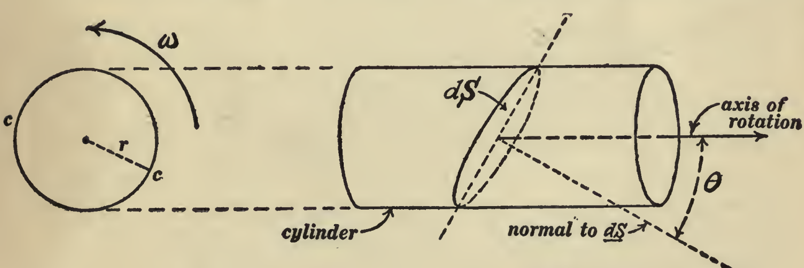


Fig. 121. Fig. 122.

the line integral divided by the area of the circle is equal to 2ω, as explained in Art. 129. Consider a cylindrical surface of which the end view is the circle cc in Fig. 121. A side view of this

cylinder is shown in Fig. 122. Let dS be the area of an oblique plane section of the cylinder. Let dL be the line integral of v' around the boundary of dS. It can be shown without difficulty that dL is equal to $2\pi r^2\omega$. But the area of dS is $\frac{\pi r^2}{\cos\theta}$. Therefore $\frac{dL}{dS}$ is equal to $2\omega\cos\theta$. That is, 2ω is a vector whose component in the direction of the normal to dS is equal to $\frac{dL}{dS}$.

136. The distortion of a small portion a body which has been twisted or bent in any manner whatever.—Consider a particle p of an unbent beam. Imagine the beam to be bent or twisted in any manner whatever, and let v be a line drawn from p to p', where p' is the position of the particle after the beam is bent. Then the line v is a vector, it is called the *displacement vector* of the particle p, and it may be thought of as *being at* p because it refers to the point p. Now v has a definite value and a definite direction at each point of the unbent beam, therefore it is a distributed vector. That is the unbent beam may be thought of as a *vector field* in its relation to the bent or twisted beam.

Consider a very small portion of the unbent beam. The displacement vector v may be thought of as having a *linear* distribution throughout this small portion of the beam as explained in Art. 133. Therefore the displacement-vector field v may be resolved into three parts as explained in Art. 133. One part v_0 is a simple translatory displacement of the entire small portion of the beam; another part v' is a simple rotation of the small portion of the beam about a definite axis; and a third part v'' consists of three mutually perpendicular stretches.* This is a fundamental proposition in the theory of elasticity.†

* The expression *continuous stretches* as used in Art. 133 referred to what takes place in a rubber band while it is being stretched, and the word *stretch* as here used refers to what has taken place in a rubber band after it has been stretched.

† See the chapter on elasticity in Franklin and MacNutt's *Mechanics and Heat*, The Macmillan Co., New York, 1910.

PROBLEM.

1. One end of a rubber band is fixed and the band is stretched by moving the other end of the band. The initial length of the band is 10 inches, the increase of length is 0.5 inch. Express the displacement of each particle of the stretched band in terms of the initial distance x of the particle from the fixed end of the band, ignoring the lateral contraction of the band. Ans. Displacement $= 0.05x$.

APPENDIX A.

PROBLEMS GROUP 1.

(After Art. 34.)

Differentiate the following:

1. $y = \dfrac{x^{n+1}}{n+1}$,
$$\frac{dy}{dx} = x^n.$$

2. $y = 4x^5 + 5x^4$,
$$\frac{dy}{dx} = 20x^3(x+1).$$

3. $y = 6x^6 - 5x^5 + 4x^4 - 3x^3$,
$$\frac{dy}{dx} = 36x^5 - 25x^4 + 16x^3 - 9x^2.$$

4. $y = x^2(x+7)$,
$$\frac{dy}{dx} = x(3x+14).$$

5. $y = 3x^3(x^2 - 5x)$,
$$\frac{dy}{dx} = 15x^3(x-4).$$

$abr^7 - acr^5 + ar^4d$
$7abr^6 - 5acr^4 + 4ar^3d$

6. $\theta = ar^4(br^3 - cr + d)$,
$$\frac{d\theta}{dr} = ar^3(7br^3 - 5cr + 4d).$$

7. $\theta = (r+3)(2r+5)$,
$$\frac{d\theta}{dr} = 4r + 11.$$

8. $\theta = (2r^2 + 3r)(3r^2 + 2r)$,
$$\frac{d\theta}{dr} = 3r(8r^2 + 13r + 4).$$

9. $\theta = (2 - r)(3 - r)$,
$$\frac{d\theta}{dr} = 2r - 5.$$

10. $\theta = (r^3 - r^2 + r)(r^2 - r + 1)$,
$$\frac{d\theta}{dr} = (r^2 - r + 1)(5r^2 - 3r + 1).$$

11. $s = (at + b)(ct + e)$,
$$\frac{ds}{dt} = 2act + ae + bc.$$

12. $s = (a - bt)(c - et)$,
$$\frac{ds}{dt} = 2bet - (ae + bc).$$

13. $s = (3t^2 - 2t - 1)(1 - 3t^2 + 5t^3)$,
$$\frac{ds}{dt} = 75t^4 - 76t^3 + 3t^2 + 12t - 2.$$

14. $s = (2 + t)^3$,
$$\frac{ds}{dt} = 3(2 + t)^2.$$

15. $s = (1 + 2x)^3$,
$$\frac{ds}{dx} = 6(1 + 2x)^2.$$

16. $s = t^2(2t + 3)^5$,
$$\frac{ds}{dt} = 2t(7t + 3)(2t + 3)^4.$$

17. $s = at^3(bt^2 + ct + e)$,
$$\frac{ds}{dt} = at^2(5bt^2 + 4ct + 3e).$$

18. $y = (1 + 2x)^3(2x^2 + 1)^4$,
$$\frac{dy}{dx} = 2(1 + 2x)^2(2x^2 + 1)^3(3 + 8x + 22x^2).$$

19. $y = (ax + b)^2(a + bx)^3$,
$$\frac{dy}{dx} = (ax + b)(a + bx)^2(5abx + 2a^2 + 3b^2).$$

20. $y = \dfrac{x}{x+2}$,

$$\frac{dy}{dx} = \frac{2}{(x+2)^2}.$$

21. $y = \dfrac{a-x}{b-x}$,

$$\frac{dy}{dx} = \frac{a-b}{(b-x)^2}.$$

22. $y = \dfrac{(x+3)^2}{(2x+5)^4}$,

$$\frac{dy}{dx} = -\frac{2(x+3)(2x+7)}{(2x+5)^5}.$$

23. $y = \dfrac{(ax+b)^3}{(cx+e)^7}$,

$$\frac{dy}{dx} = \frac{(ax+b)^2(3ae - 7bc - 4acx)}{(cx+e)^8}.$$

24. $y = \left(\dfrac{1+x}{2x+3}\right)^3$,

$$\frac{dy}{dx} = \frac{3(1+x)^2}{(2x+3)^4}.$$

25. $y = \left(\dfrac{2x-1}{x-2}\right)^4$,

$$\frac{dy}{dx} = -\frac{12(2x-1)^3}{(x-2)^5}.$$

26. $y = \left(\dfrac{ax^2 - bx + c}{ex}\right)^5$,

$$\frac{dy}{dx} = \frac{5(ax^2 - c)(ax^2 - bx + c)}{e^5x^6}.$$

27. Find the slope of the curve $y = x^3 - 2x^2 + 3$ at (*a*) $x = 0$, (*b*) $x = 2$, and (*c*) $x = -2$. Ans. (*a*) 0, (*b*) 4, (*c*) -20.

28. At what angles does the curve $y = x(x-2)(x-3)$ cut the x-axis? Ans. (*a*) At $x = 0$, slope is 6, (*b*) at $x = 2$, slope is -2, (*c*) at $x = 3$, slope is 3.

29. Is $\dfrac{2x+3}{4x+5}$ an increasing or a decreasing function of x? Ans. Decreasing.

30. Is $\dfrac{5x+2}{3x+4}$ an increasing or a decreasing function of x? Ans. Increasing.

31. When $y = 2x^3 - 6x + 5$, for what values of x is y (*a*) an increasing function, and (*b*) a decreasing function? Ans. (*a*) All values of x less than -1 or greater than $+1$, (*b*) all values of x between -1 and $+1$.

PROBLEMS GROUP 2.

(*After Art. 39.*)

Differentiate the following functions

Note.—The forms referred to in the answers to the following problems are found in the table of integrals, see Appendix B. Thus the answer to problem 4 is given by form 45, as

$$\frac{dy}{dx} = \frac{1}{\sqrt{a+bx}}$$

the answer to problem 5 is found to be

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+a^2}}, \text{ etc.}$$

1. $y = x^2 - 3x + \dfrac{3}{x} - \dfrac{7}{x^2}$, $\qquad \dfrac{dy}{dx} = 2x - 3 - \dfrac{3}{x^2} - \dfrac{14}{x^3}$.

2. $y = 3\sqrt[3]{x^2} - 2\sqrt{x^3}$, $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sqrt[3]{x}} - 3\sqrt{x}$.

3. $y = \dfrac{2\sqrt[4]{x^3}}{5} - \dfrac{5}{3\sqrt[3]{x^4}}$, $\qquad \dfrac{dy}{dx} = \dfrac{3}{10}x^{-\frac{1}{4}} + \dfrac{20}{9}x^{-\frac{7}{3}}$.

4. $y = \dfrac{2\sqrt{a+bx}}{b}$, Form 45.

5. $y = \sqrt{x^2+a^2}$, Form 61.

6. $y = -\sqrt{a^2-x^2}$, Form 53.

7. $\theta = (r^3 - a^3)^{\frac{5}{3}}$, $\qquad \dfrac{d\theta}{dr} = 5r^2(r^3 - a^3)^{\frac{2}{3}}$.

8. $y = \frac{2}{3}b \sqrt{(a + bx)^3}$, Form 42.

9. $y = \frac{1}{3} \sqrt{(x^2 + a^2)^3}$, Form 58.

10. $y = - \frac{1}{3} \sqrt{(a^2 - x^2)^3}$, Form 49.

11. $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$, $\qquad \dfrac{dy}{dx} = - \dfrac{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}}$.

12. $\theta = r^3 \sqrt{r - r^2}$, $\qquad \dfrac{d\theta}{dr} = \dfrac{r^3(7 - 8r)}{2 \sqrt{r - r^2}}$.

13. $\theta = (r - a)^{\frac{1}{2}}(r + a)^{\frac{3}{2}}$, $\qquad \dfrac{d\theta}{dr} = \dfrac{(2r - a) \sqrt{r + a}}{\sqrt{r - a}}$.

14. $y = \dfrac{2(2a - bx) \sqrt{a + bx}}{3b^2}$, Form 46.

15. $y = \dfrac{2(2a - 3bx) \sqrt{(a + bx)^3}}{15b^2}$, Form 43.

16. $y = \dfrac{2(8a^2 - 4abx + 3b^2x^2) \sqrt{a + bx}}{15b^3}$, Form 47.

17. $y = \dfrac{2(8a^2 - 12abx + 15b^2x^2) \sqrt{(a + bx)^3}}{105b^3}$, Form 44.

18. $y = \dfrac{1}{b(a + bx)}$, Form 34.

19. $y = - \dfrac{\sqrt{x^2 + a^2}}{a^2x}$, Form 64.

20. $y = \dfrac{x}{a^2 \sqrt{x^2 + a^2}}$, Form 66.

21. $y = \dfrac{x}{a^2 \sqrt{a^2 - x^2}}$, Form 56.

22. $y = - \dfrac{\sqrt{2ax - x^2}}{ax}$, Form 73.

23. $y = \dfrac{(x^2 - a^2)^{\frac{4}{3}}}{x^{\frac{3}{2}}}$, $\qquad \dfrac{dy}{dx} = \dfrac{(x^2 - a^2)^{\frac{1}{3}}(7x^2 + 9a^2)}{6x^{\frac{5}{2}}}$.

24. $\theta = \sqrt[3]{\dfrac{a^2 - x^2}{a^2 + x^2}}$, $\qquad \dfrac{d\theta}{dx} = \dfrac{- 4a^2x}{3(a^2 - x^2)^{\frac{2}{3}}(a^2 + x^2)^{\frac{4}{3}}}$.

25. $y = \log_a x^n, \qquad \dfrac{dy}{dx} = \dfrac{n \log_a e}{x}$

26. $u = \log (3v^2 + 4v^3)^3, \qquad \dfrac{du}{dv} = \dfrac{18(1 + 2v)}{v(3 + 4v)}.$

27. $y = \dfrac{1}{b} \log (a + bx),$ Form 33.

28. $y = \dfrac{1}{2b} \log \left(x^2 + \dfrac{a}{b}\right),$ Form 39.

29. $y = \log (x + \sqrt{1 + x^2}),$ Form 10.

30. $y = \log (x + \sqrt{x^2 - 1}),$ Form 11.

31. $y = \log (x + \sqrt{x^2 + a^2}),$ Form 60.

32. $y = -\log\left(\dfrac{1}{x} + \sqrt{\dfrac{1}{x^2} + 1}\right),$ From 16.

33. $y = -\log \left(\dfrac{1}{x} + \sqrt{\dfrac{1}{x^2} - 1}\right),$ Form 15.

34. $s = [\log (4t^3 + 3)]^4, \qquad \dfrac{ds}{dt} = \dfrac{48t^2[\log (4t^3 + 3)]^3}{4t^3 + 3}.$

35. $s = t^{\frac{3}{2}} \log (at^2 - bt)^{\frac{2}{3}} \qquad \dfrac{ds}{dt} = \dfrac{2t^{\frac{1}{2}}(2at - b)}{3(at - b)} + t^{\frac{1}{2}} \log (at^2 - bt).$

36. $y = \dfrac{1}{2c} \log \dfrac{c + x}{c - x},$ Form 38.

37. $y = \frac{1}{2} \log \dfrac{1 + x}{1 - x},$ Form 13.

38. $y = \dfrac{1}{a} \log \dfrac{x}{a + \sqrt{x^2 + a^2}},$ Form 63.

39. $y = \dfrac{1}{a} \log \dfrac{\sqrt{x^2 + a^2} - a}{x},$ Form 63.

40. $y = \dfrac{1}{\sqrt{b^2 - 4ac}} \log \dfrac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}},$ Form 41.

41. $y = \dfrac{1}{b^2} \left[a + bx - a \log (a + bx)\right],$ Form 35.

42. $y = \dfrac{1}{b^2} \left[\log (a + bx) + \dfrac{a}{a + bx}\right],$ Form 36.

43. $y = \sqrt{a^2 - x^2} - a \log \dfrac{a + \sqrt{a^2 - x^2}}{x}$, Form 51.

44. $y = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2} \log (x + \sqrt{x^2 + a^2})$, Form 57.

45. $y = \dfrac{x}{2}\sqrt{x^2 + a^2} - \dfrac{a^2}{2} \log (x + \sqrt{x^2 + a^2})$, Form 62.

46. $y = \dfrac{x}{8}(2x^2 + 5a^2)\sqrt{x^2 + a^2} + \dfrac{3a^4}{8} \log (x + \sqrt{x^2 + a^2})$, Form 65.

47. $y = \dfrac{x}{8}(2x^2 + a^2)\sqrt{x^2 + a^2} - \dfrac{a^4}{8} \log (x + \sqrt{x^2 + a^2})$, Form 59.

48. $y = \log_{10} (3x + 2)$, $\dfrac{dy}{dx} = \dfrac{1.3029}{3x + 2}$

49. $y = \dfrac{1}{x \log x}$, $\dfrac{dy}{dx} = -\dfrac{1 + \log x}{(x \log x)^2}$.

50. $y = x^{\log a}$, $\dfrac{dy}{dx} = \log a \cdot x^{\log a - 1}$

51. $y = a^{x^2}$, $\dfrac{dy}{dx} = 2xa^{x^2} \log a$.

52. $y = a^{\log x}$, $\dfrac{dy}{dx} = \dfrac{\log a \cdot a^{\log x}}{x}$.

53. $\theta = c^x e^x$, $\dfrac{d\theta}{dx} = c^x e^x (1 + \log c)$.

54. $y = e^x x^e$, $\dfrac{dy}{dx} = e^x x^{e-1}(e + x)$.

55. $y = 4^x x^4$, $\dfrac{dy}{dx} = 4^x x^3 (4 + x \log 4)$.

56. $y = e^{\frac{x^2+2}{x}}$, $\dfrac{dy}{dx} = \left(1 - \dfrac{2}{x^2}\right) e^{\frac{x^2+2}{x}}$.

57. $y = b^{\frac{2-x^3}{x^2}}$, $\dfrac{dy}{dx} = -\, b^{\frac{2-x^3}{x^2}} \cdot \left(1 + \dfrac{4}{x^3}\right) \log b$.

58. $y = \dfrac{5x + 3}{x - 3} e^{-2x}$, $\dfrac{dy}{dx} = \dfrac{24x - 10x^2}{(x - 3)^2} e^{-2x}$

59. $y = (e^{3x} - 2)^{\frac{2}{3}}$, $\dfrac{dy}{dx} = \dfrac{2e^{3x}}{(e^{3x} - 2)^{\frac{1}{3}}}$.

60. $y = \log \log x - \frac{1}{\log x}$, $\quad \frac{dy}{dx} = \frac{1 + \log x}{x(\log x)^2}$.

61. $y = \frac{c}{2}(e^{2x} - e^{-2x})$, $\quad \frac{dy}{dx} = c(e^{2x} + e^{-2x})$.

62. $s = (2e^{2t} - 1)(1 - 3e^{-3t})$, $\quad \frac{ds}{dt} = 4e^{2t} + 6e^{-t} - 9e^{-3t}$.

63. $y = \log \frac{\log x}{1 + \log x}$, $\quad \frac{dy}{dx} = \frac{1}{x \log x\,(1 + \log x)}$.

64. $y = e^{\log_a x} - x^{\log_a e}$, $\quad \frac{dy}{dx} = \frac{\log_a e}{x}\left(a^{\log_a x} - x^{\log_a e}\right)$.

65. $y = \frac{e^x + x^a}{e^x - x^a}$, $\quad \frac{dy}{dx} = \frac{2e^x x^{a-1}(a - x)}{(e^x - x^a)^2}$.

66. $y = x[(\log x)^2 - 2 \log x + 2]$, $\quad \frac{dy}{dx} = (\log x)^2$.

67. $y = \log \frac{e^{nx} - 1 + e^{-nx}}{e^{nx} + 1 + e^{-nx}}$, $\quad \frac{dy}{dx} = \frac{2n(e^{nx} - e^{-nx})}{e^{2nx} + 1 + e^{-2nx}}$.

68. $y = \frac{\log (e^{nx} - 1)}{\log (e^{nx} + 1)}$, $\quad \frac{dy}{dx} = \frac{ne^{nx}\left[\frac{\log (e^{nx} + 1)}{e^{nx} - 1} - \frac{\log (e^{nx} - 1)}{e^{nx} + 1}\right]}{[\log (e^{nx} + 1)]^2}$.

PROBLEMS GROUP 3.

(*After Art. 42.*)

Differentiate the following:

1. $y = \sin 5x$, $\quad \frac{dy}{dx} = 5 \cos 5x$.

2. $y = \cos nx$, $\quad \frac{dy}{dx} = -n \sin nx$.

3. $y = \sin 3x^2$, $\quad \frac{dy}{dx} = 6x \cos 3x^2$.

4. $y = \frac{x}{2} + \frac{1}{4} \sin 2x$, Form 21.

5. $y = \frac{x}{2} - \frac{1}{4} \sin 2x$, Form 20.

6. $y = \sin 3x \cos 2x + \cos 3x \sin 2x, \quad \dfrac{dy}{dx} = 5 \cos 5x.$

7. $y = \dfrac{\sin (m-n)x}{2(m-n)} + \dfrac{\sin (m+n)x}{2(m+n)},$ Form 26.

8. $y = \dfrac{\sin (m-n)x}{2(m-n)} - \dfrac{\sin (m+n)x}{2(m+n)},$ Form 24.

9. $y = \dfrac{\cos (m-n)x}{2(m-n)} + \dfrac{\cos (m+n)x}{2(m+n)},$ Form 25.

10. $y = \log \sin x,$ Form 8.

11. $y = -\log \cos x,$ Form 7.

12. $s = \sin(t^2 - 3t + 2), \quad \dfrac{ds}{dt} = (2t-3)\cos(t^2 - 3t + 2).$

13. $s = \log \cos (1 - t^2), \quad \dfrac{ds}{dt} = 2t \tan (1 - t^2).$

14. $y = \log(a \sin^2 x + b \cos^2 x), \quad \dfrac{dy}{dx} = \dfrac{2(a-b)\tan x}{a \tan^2 x + b}.$

15. $y = 2x^2 \sin 2x + 2x \cos 2x - \sin 2x, \quad \dfrac{dy}{dx} = 4x^2 \cos 2x.$

16. $y = e^x \sin (2 - x^2), \quad \dfrac{dy}{dx} = e^x [\sin (2-x^2) - 2x \cos (2-x^2)].$

17. $r = \theta^2 \cos (e^\theta - e^{-\theta}), \quad \dfrac{dr}{d\theta} = \theta[2 \cos (e^\theta - e^{-\theta}) - \theta (e^\theta + e^{-\theta}) \sin (e^\theta - e^{-\theta})].$

18. $y = \log \dfrac{\cos x}{\cos (x+c)}, \quad \dfrac{dy}{dx} = \dfrac{\sin c}{\cos x \cos (x+c)}.$

19. $y = \sin^3 2x \cos^2 3x, \quad \dfrac{dy}{dx} = 6 \sin^2 2x \cos 3x \cos 5x.$

20. $y = e^{\sin x^2} \quad \dfrac{dy}{dx} = 2x \cos x^2 \cdot e^{\sin x^2}$

21. $y = \log \dfrac{\sin x - \cos x}{\sin x + \cos x}, \quad \dfrac{dy}{dx} = -\dfrac{2}{\cos 2x}.$

22. $y = \log \dfrac{\sin \frac{1}{2}(x - \alpha)}{\sin \frac{1}{2}(x + \alpha)}, \quad \dfrac{dy}{dx} = \dfrac{\sin \alpha}{2 \sin \frac{1}{2}(x-\alpha) \sin \frac{1}{2}(x+\alpha)}.$

23. The lengths of crank radius and connecting rod of a steam engine are 2 feet and 10 feet respectively, as shown in Fig. $p23$, and the crank revolves at the uniform speed of 2 revolutions per second. Find the velocity of the piston for the following values of the angle θ: (*a*) $\theta = 0$, (*b*) $\theta = 45°$, (*c*) $\theta = 90°$ and (*d*) $\theta = 135°$. Ans. (*a*) zero, (*b*) 20.32 feet per second, (*c*) 25.13 feet per second, (*d*) 15.24 feet per second. Fig. $p23$ is $1\frac{1}{8}''$ high $\times$ $4''$ long.

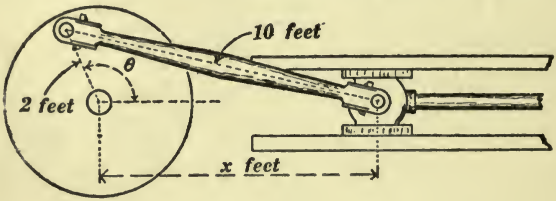


Fig. $p23$.

PROBLEMS GROUP 4.

(*After Art. 43.*)

Differentiate the following:

1. $y = \cot x$, $\qquad \frac{dy}{dx} = -\operatorname{cosec}^2 x.$

2. $y = \sec x$, $\qquad \frac{dy}{dx} = \sec x \tan x.$

3. $y = \operatorname{cosec} x$, $\qquad \frac{dy}{dx} = -\operatorname{cosec} x \cot x.$

4. $y = \operatorname{vers} x$, $\qquad \frac{dy}{dx} = \sin x.$

5. $y = \tan (x^2 - 2x)$, $\qquad \frac{dy}{dx} = 2(x - 1) \sec^2 (x^2 - 2x).$

6. $y = \sec^n nx$, $\qquad \frac{dy}{dx} = n^2 \sec^n nx \tan nx.$

7. $y = \log \tan \frac{x}{2}$, $\qquad$ Form 19.

8. $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, Form 18.

9. $y = \log(\sec x + \tan x)$, Form 18.

10. $y = \log(\operatorname{cosec} x - \cot x)$, Form 19.

11. $y = e^{2x} \operatorname{cosec} 2x$, $\quad \frac{dy}{dx} = 2e^{2x} \operatorname{cosec} 2x(1 - \cot 2x)$.

12. $y = (\tan u - 3 \cot u)\sqrt{\tan u}$, $\quad \frac{dy}{du} = \frac{3 \sec^4 u}{2 \tan^{\frac{3}{2}} u}$.

13. $s = t^n \operatorname{cosec}^m nt$, $\quad \frac{ds}{dt} = nt^{n-1} \operatorname{cosec}^m nt(1 - mt \cos nt)$.

14. $y = \frac{2 \tan \theta}{1 - \tan^2 \theta}$, $\quad \frac{dy}{d\theta} = \frac{2 \sec^4 \theta}{(1 - \tan^2 \theta)^2}$.

15. $y = \log [\tan (e^x + 3)^{\frac{1}{3}}]^3$, $\quad \frac{dy}{dx} = \frac{2e^x(e^x + 3)^{-\frac{2}{3}}}{\sin [2(e^x + 3)^{\frac{1}{3}}]}$.

16. $\theta = \log \frac{\tan \frac{r}{2} - 2}{2 \tan \frac{r}{2} - 1}$, $\quad \frac{d\theta}{dr} = \frac{3}{4 - 5 \sin r}$.

17. $r = \frac{a \sin \theta + b \operatorname{vers} \theta}{a \sin \theta - b \operatorname{vers} \theta}$ $\quad \frac{dr}{d\theta} = \frac{2ab \operatorname{vers} \theta}{(a \sin \theta - b \operatorname{vers} \theta)^2}$.

PROBLEMS GROUP 5.

(After Art. 44.)

Differentiate the following:

1. $y = \cos^{-1} x$, $\quad \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$.

2. $y = \tan^{-1} x$, Form 12.

3. $y = \cot^{-1} x$, $\quad \frac{dy}{dx} = -\frac{1}{1 + x^2}$.

4. $y = \sec^{-1} x$, Form 14.

5. $y = \operatorname{cosec}^{-1} x$, $\quad \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2 - 1}}$.

6. $y = \operatorname{vers}^{-1} x$, Form 17.

7. $y = \sin^{-1}\frac{x}{a}$, Form 52.

8. $y = \frac{1}{c}\tan^{-1}\frac{x}{c}$, Form 37.

9. $y = \frac{2}{\sqrt{4ac - b^2}}\tan^{-1}\frac{2cx + b}{\sqrt{4ac - b^2}}$, Form 40.

10. $y = \frac{1}{a}\sec^{-1}\frac{x}{a}$, Form 67.

11. $y = \text{vers}^{-1}\frac{x}{a}$, Form 70.

12. $y = x\sin^{-1}x + \sqrt{1 - x^2}$, Form 27.

13. $y = x\cos^{-1}x - \sqrt{1 - x^2}$, Form 28.

14. $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$, Form 48.

15. $y = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$, Form 54.

16. $y = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$, Form 50.

17. $y = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\sin^{-1}\frac{x}{a}$, Form 55.

18. $\omega = \cos^{-1}\sqrt{\text{vers}\, t}$ $\qquad \frac{d\omega}{dt} = -\frac{1}{2}\sqrt{1 + \sec t}$.

19. $r = \tan^{-1}\frac{3\theta - 2}{5} + \cot^{-1}\frac{3\theta - 12}{6\theta + 1}$, $\qquad \frac{dr}{d\theta} = 0$.

20. $r = \cos^{-1}\frac{\theta - 3}{3} - 2\sqrt{\frac{6 - \theta}{\theta}}$, $\qquad \frac{dr}{d\theta} = \sqrt{\frac{6}{\theta^3} - \frac{1}{\theta^2}}$.

21. $s = t^2\sec^{-1}\frac{t}{2} - 2\sqrt{t^2 - 4}$, $\qquad \frac{ds}{dt} = 2t\sec^{-1}\frac{t}{2}$.

22. $\omega = \sqrt{2}\,[\sec^{-1}(\sec t + \tan t)]$, $\qquad \frac{d\omega}{dt} = \frac{\sec t}{\sqrt{(\sec t + \tan t)\tan t}}$.

23. $y = \tan^{-1}\frac{4 + 5\tan x}{3}$ $\qquad \frac{dy}{dx} = \frac{3}{5 + 4\sin 2x}$.

24. $s = \tan^{-1}\frac{e^t + e^{-t}}{e^t - e^{-t}}$, $\qquad \frac{ds}{dt} = \frac{-2}{e^{2t} + e^{-2t}}$.

25. $y = \tan^{-1}\left(\frac{a}{b}\tan x\right) - \tan^{-1}\frac{b}{a}$, $\frac{dy}{dx} = \frac{ab}{a^2 \sin^2 x + b^2 \cos^2 x}$.

26. $y = \tan^{-1}\frac{a^2 \tan x - b^2}{ab(1 + \tan x)}$, $\frac{dy}{dx} = \frac{ab}{a^2 \sin^2 x + b^2 \cos^2 x}$.

27. $y = \sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a}$, Form 72.

28. $y = -\sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a}$, Form 71.

29. $y = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}$, Form 68.

30. $y = -\frac{3a^2 + ax - 2x^2}{6}\sqrt{2ax - x^2} + \frac{a^3}{2}\operatorname{vers}^{-1}\frac{x}{a}$, Form 69.

PROBLEMS GROUP 6.

(After Art. 45.)

Differentiate the following:

1. $y = x^{nx}$, $\frac{dy}{dx} = nx^{nx}(1 + \log x)$.

2. $\theta = (1 - t)^{1-t}$, $\frac{d\theta}{dt} = -(1 - t)^{1-t}[1 + \log(1 - t)]$

3. $s = 3t^{t^3}$, $\frac{ds}{dt} = 3t^{t^3+2}[1 + 3\log t]$.

4. $y = x^{(\log x)^n}$, $\frac{dy}{dx} = (n + 1)(\log x)^n x^{(\log x)^n - 1}$

5. $y = (\log x)^{\log x}$ $\frac{dy}{dx} = \frac{(\log x)^{\log x}}{x}\Big[1 + \log(\log x)\Big]$.

6. $y = (3x^2 + 1)^{2(x+2)}$, $\frac{dy}{dx} =$

$$(3x^2 + 1)^{2(x+2)}\left\{2\log(3x^2 + 1) + \frac{12x(x + 2)}{3x^2 + 1}\right\}.$$

PROBLEMS GROUP 7.

(After rule III of Art. 57.)

Integrate the following differential equations.

1. $dy = \left(3x^7 - 7x^3 + \frac{3}{x} - \frac{7}{x^2}\right)dx,\quad y = \frac{3x^8}{8} - \frac{7x^4}{4} + \log x^3 + \frac{7}{x} + C.$

2. $dy = (2 - x^2)^2 x^2 \cdot dx,\qquad y = \frac{4x^3}{3} - \frac{4x^5}{5} + \frac{x^7}{7} + C.$

3. $dy = 3(2x^3 + 3)x^2 \cdot dx,\qquad y = \frac{(2x^3 + 3)^2}{4} + C.$

4. $dy = \frac{dx}{a + bx},$ Form 33.

5. $dy = \frac{dx}{(a + bx)^2},$ Form 34.

6. $dy = \frac{x \cdot dx}{a + bx^2},$ Form 39.

7. $dy = \sqrt{a + bx} \cdot dx,$ Form 42.

8. $dy = \frac{dx}{\sqrt{a + bx}},$ Form 45.

9. $dy = x\sqrt{a^2 - x^2} \cdot dx$ Form 49.

10. $dy = \frac{x \cdot dx}{\sqrt{a^2 - x^2}},$ Form 53.

11. $dy = \frac{dx}{x\sqrt{2ax - x^2}},$ Form 73.

12. $dy = a^x \cdot dx,$ Form 4.

13. $dy = (e^{4x} + a^{5x} + 3b^{-2x})dx,\quad y = \frac{e^{4x}}{4} + \frac{a^{5x}}{5 \log a} - \frac{3b^{-2x}}{2 \log b} + C.$

14. $dy = \frac{(e^x - 1)^2}{\sqrt[3]{e^{2x}}}dx,\qquad y = \frac{3e^{\frac{4x}{3}}}{4} - 6e^{\frac{x}{3}} - \frac{3e^{-\frac{2x}{3}}}{2} + C.$

15. $dy = \sin x \cos x \cdot dx \qquad y = \frac{\sin^2 x}{2} + C.$

16. $dy = \tan x \cdot dx,$ Form 7.

Note.—Take $\tan x = \frac{\sin x}{\cos x}.$

17. $dy = \cot x \cdot dx,$ Form 8.

18. $dy = \sin^5 x dx,$ $y = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C.$

Note.—Take $\sin^5 x = (1 - \cos^2 x)^2 \sin x.$

19. $dy = \sin^4 x \cos^5 x \cdot dx,$ $y = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C.$

20. $dy = \sin^2 x \cdot dx,$ Form 20.

Note.—Put $\sin^2 x = \frac{1 - \cos 2x}{2}.$

21. $dy = \cos^2 x \cdot dx,$ Form 21.

22. $dy = \cos^6 x \cdot dx,$ $y = \frac{5x}{16} + \frac{\sin 2x}{4} - \frac{\sin^3 2x}{48} + \frac{3\sin 4x}{64} + C.$

23. $dy = \sin mx \sin nx \cdot dx,$ Form 24.

Note.—Put $\sin mx \sin nx = \frac{1}{2}\cos(m - n)x - \frac{1}{2}\cos(m + n)x.$

24. $dy = \tan^5 x \cdot dx,$ $y = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \log \cos x + C.$

Note.—Put $\tan^5 x = \tan^3 x (\sec^2 x - 1).$

25. $dy = \cot^4 x \cdot dx,$ $y = x + \cot x - \frac{\cot^3 x}{3} + C.$

26. $dy = \operatorname{cosec}^6 x \cdot dx,$ $y = C - \cot x - \frac{2\cot^3 x}{3} - \frac{\cot^5 x}{5}.$

Note.—Put $\operatorname{cosec}^4 x = (1 + \cot^2 x)^2.$

27. $dy = \tan^3 x \sec^6 x \cdot dx,$ $y = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{3} + \frac{\tan^8 x}{8} + C.$

Note.—Put $\sec^4 x = (1 + \tan^2 x)^2.$

28. $dy = \tan^5 x \sec^3 x \cdot dx,$ $y = \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C.$

Note.—Put $\tan^4 x = (\sec^2 x - 1)^2$ and write $\tan^5 x \sec^3 x = (\sec^2 x - 1)^2$ $\sec^2 x \cdot \sec x \tan x \cdot dx.$

29. $dy = \frac{dx}{\sqrt{4 - 25x^2}},$ $y = \frac{1}{5}\sin^{-1}\frac{5x}{2} + C.$

30. $dy = \frac{dx}{\sqrt{25x^2 - 4}},$ $y = \frac{1}{5}\log(5x + \sqrt{25x^2 - 4}) + C.$

31. $dy = \dfrac{dx}{4 + 25x^2}$, $\qquad y = \dfrac{1}{2}\tan^{-1}\dfrac{5x}{2} + C.$

32. $dy = \dfrac{dx}{4 - 25x^2}$, $\qquad y = \dfrac{1}{4}\log\dfrac{2 + 5x}{2 - 5x} + C.$

33. $dy = \dfrac{dx}{x\sqrt{9x^2 - 4}}$, $\qquad y = \dfrac{1}{2}\sec^{-1}\dfrac{3x}{2} + C.$

34. $dy = \dfrac{dx}{\sqrt{4x - x^2}}$, $\qquad y = \text{vers}^{-1}\dfrac{x}{2} + C.$

35. $dy = \dfrac{x + 3}{4x^2 - 5}\cdot dx,\quad y = \dfrac{1}{8}\log(4x^2 - 5) + \dfrac{3}{4\sqrt{5}}\log\dfrac{2x - \sqrt{5}}{2x + \sqrt{5}} + C.$

36. Find the length of the arc of the parabola in Fig. 13 on page 24 from $x = 0$ to $x = 10$ inches, the value of k being 2 when x and y are both expressed in inches.

Ans. 201 square inches.

PROBLEMS GROUP 8.

(After rule IV of Art. 57.)

Verify the following:

1. $\int x \sin x \cdot dx = -x\cos x + \sin x + C.$

Note.—Use rule IV. Try $u = \sin x$, $dv = x\cdot dx$; also try $u = x$, $dv = \sin x\cdot dx$.

2. $\int x\cos x \cdot dx = x\sin x + \cos x + C.$

3. $\int x^3 \log x \cdot dx = \dfrac{x^4}{4}(\log x - \tfrac{1}{4}) + C.$

4. $\int x(e^{3x} + e^{-3x})dx = \dfrac{x}{3}(e^{3x} - e^{-3x}) - \tfrac{1}{9}(e^{3x} + e^{-3x}) + C.$

5. $\int \log(ax + b)\cdot dx = \dfrac{1}{a}(ax + b)\log(ax + b) - x + C.$

6. $\int x^3 \sin x\cdot dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$

Note.—Applying rule IV, taking $u = x^3$ and $dv = \sin x\cdot dx$, we get an expression involving $\int x^2 \cos x\cdot dx$. This expression can again be reduced by

applying rule IV (to the part to be integrated), making $u = x^2$ and $dv = \cos x \cdot dx$; and so on.

7. $\int x^3 \sin 2x \cdot dx = \frac{3}{8}(2x^2 - 1) \sin 2x - \frac{1}{4}(2x^3 - 3x)\cos 2x + C.$

8. $\int e^{ax} \sin bx \cdot dx = \dfrac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$

Note.—Integrate by parts, taking $u = e^{ax}$, then:

$$\int e^{ax} \sin bx \cdot dx = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx \cdot dx \qquad (1)$$

Integrate by parts, taking $u = \sin bx$, then:

$$\int e^{ax} \sin bx \cdot dx = \frac{e^{ax} \sin bx}{a} - \frac{b}{a}\int e^{ax} \cos bx \cdot dx \qquad (2)$$

Take the sum of equation (1) multiplied by b^2 and equation (2) multiplied by a^2 in order to eliminate the term containing $\int e^{ax} \cos bx \cdot dx$ and the result given above is obtained. By subtracting equation (2) from equation (1), $\int e^{ax} \sin bx \cdot dx$ can be eliminated and the value of $\int e^{ax} \cos bx \cdot dx$ obtained.

9. $\int e^{3x} \cos 5x \cdot dx = \dfrac{e^{3x}}{34}(5 \sin 5x + 3 \cos 5x) + C.$

10. $\int e^{-2x} \sin x \cdot dx = -\dfrac{e^{-2x}}{5}(2 \sin x + \cos x) + C.$

11. $\int e^{-\frac{x}{2}} \cos \dfrac{x}{3} \cdot dx = \dfrac{6e^{-\frac{x}{2}}}{13}\left(2 \sin \dfrac{x}{3} - 3 \cos \dfrac{x}{3}\right) + C.$

PROBLEMS GROUP 9.

(*Special methods a of Art. 57.*)

Verify the following:

1. $\displaystyle\int \frac{5x + 4}{x^2 + x - 2} \cdot dx = \int \frac{3 \cdot dx}{x - 1} + \int \frac{2 \cdot dx}{x + 2}$
$\quad = 3 \log (x - 1) + 2 \log (x + 2) + C.$

Note.—A full discussion of the integration of rational fractions is given in Byerly's *Integral Calculus*. See note on page 86.

In this particular case the denominator is equal to $(x - 1)(x + 2)$ and

$\frac{5x+4}{x^2+x-2}$ may be broken up into what are called partial fractions as follows:

$$\frac{5x+4}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

Clear of fractions and equate coefficients of like powers of x and we find $A = 3$ and $B = 2$.

2. $$\int \frac{2x+5}{x^2+5x+4} \cdot dx = \log(x+4) + \log(x+1) + C.$$

3. $$\int \frac{x^2+x-1}{x^3+x^2-6x} \cdot dx = \tfrac{1}{6}\log x + \tfrac{1}{3}\log(x+3) + \tfrac{1}{2}\log(x-2) + C.$$

4. $$\int \frac{5x^3+1}{x^2-3x+2} \cdot dx = \int\left(5x + 15 + \frac{35x-29}{x^2-3x+2}\right)dx$$
$$= \frac{5x^2}{2} + 15x - 6\log(x-1) + 41\log(x-2) + C.$$

Note.—Whenever the degree of the numerator of a rational fraction is greater than or equal to the degree of the denominator, the fraction should be reduced to a mixed quantity.

5. $$\int \frac{x^3}{x^3-7x-6} \cdot dx = x + \frac{27}{20}\log(x-3) - \frac{8}{5}\log(x+2) + \frac{1}{4}\log(x+1) + C.$$

6. $$\int \frac{x^2}{(x-1)^2(x^2+1)} \cdot dx = \frac{1}{2}\log(x-1) - \frac{1}{2(x-1)} - \frac{1}{4}\log(x^2+1) + C.$$

Note.—$\frac{x^2}{(x-1)^2(x^2+1)}$ is broken into partial fractions by putting it equal to $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$.

7. $$\int \frac{x^2}{x^4-x^2-12} \cdot dx = \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + \frac{1}{7}\log\frac{x-2}{x+2} + C.$$

8. $$\int \frac{32}{x(x^2-4)^2} \cdot dx = -\frac{4}{x^2-4} + \log\frac{x^2}{x^2-4} + C.$$

PROBLEMS GROUP 10.

(Special methods b of Art. 57.)

Integrate the following and for answers see specified forms in the table of integrals of Appendix B.

1. $\int \sqrt{a^2 - x^2} \cdot dx$ Form 48.

Note.—Let $x = a \sin \theta$, then $dx = a \cos \theta \cdot d\theta$; and we get:

$$\int \sqrt{a^2 - x^2} \cdot dx = \int \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta \cdot d\theta$$
$$= a^2 \int \cos^2 \theta \cdot d\theta$$
$$= a^2 \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta\right]$$
$$= a^2 \left[\frac{1}{2} \sin^{-1} \frac{x}{a} + \frac{1}{2} \frac{x\sqrt{a^2 - x^2}}{a^2}\right]$$
$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2}$$

2. $\int x^2 \sqrt{a^2 - x^2} \cdot dx,$ Form 50.

3. $\int \frac{\sqrt{a^2 - x^2}}{x} \cdot dx,$ Form 51.

4. $\int (a^2 - x^2)^{\frac{3}{2}} \cdot dx,$ Form 55.

5. $\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}},$ Form 56.

6. $\int \sqrt{x^2 + a^2} \cdot dx,$ Form 57.

7. $\int x^2 \sqrt{x^2 + a^2} \cdot dx,$ Form 59.

8. $\int \frac{x^2 \cdot dx}{\sqrt{x^2 + a^2}},$ Form 62.

9. $\int \frac{dx}{x^2 \sqrt{x^2 - a^2}},$ $y = \frac{\sqrt{x^2 - a^2}}{a^2 x^2},$ Form 64.

10. $\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}},$ $y = -\frac{x}{a^2 \sqrt{x^2 - a^2}},$ Form 66.

PROBLEMS GROUP 11.

(Special methods c of Art. 57.)

Verify the following:

1. $$\int \frac{x^3 \cdot dx}{(x+a)^m} = \frac{(x+a)^{4-m}}{4-m} - 3a\frac{(x+a)^{3-m}}{3-m} + 3a^2\frac{(x+a)^{2-m}}{2-m} - a^3\frac{(x+a)^{1-m}}{1-m} + C.$$

Note.—Let $x + a = y, \quad dx = dy.$

2. $$\int \frac{x \cdot dx}{(a+bx)^{\frac{1}{3}}} = \frac{3}{5b^2}(a+bx)^{\frac{5}{3}} - \frac{3}{2b^2}(a+bx)^{\frac{2}{3}} + C.$$

Note.—Let $a + bx = y, \quad b \cdot dx = dy.$

3. $$\int \frac{dx}{e^x + e^{-x}} = \tan^{-1} e^x + C.$$

Note.—Let $e^x = y,\ x = \log y,\ dx = \frac{dy}{y}.$

4. $$\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} dx = \frac{x(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}}{6a^2} - \frac{x^{\frac{1}{3}}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}(a^{\frac{2}{3}} - 2x^{\frac{2}{3}})}{16a^{\frac{4}{3}}} + \frac{1}{16}\sin^{-1}\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}} + C.$$

Note.—Let $x = a \sin^3 \theta.$

APPENDIX B.

TABLE OF INTEGRALS.

A fairly extensive table of integrals is *A table of integrals,* arranged by B. O. Peirce and published by Ginn & Co., Boston, 1890; revised edition 1899.

Explanatory Notes.

To say that $\int 2x \cdot dx$ is x^2, is exactly the same thing as to say that the differential of x^2 is $2x \cdot dx$.

Constants of integration are omitted from this table.

Any of the integrals given in this table can be verified by differentiation. For example, we can differentiate $\left(\frac{x^{n+1}}{n+1} + C\right)$ according to Art. 30, and we get $x^n \cdot dx$; which shows that $\left(\frac{n^{n+1}}{n+1} + \text{a constant}\right)$ is the integral of $x^n \cdot dx$.

One has frequent occasion to use the following formulas: $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, and $\operatorname{vers} x = 1 - \cos x$.

The definitions of the hyperbolic functions $\sinh x$, $\cosh x$, etc., are given in Chapter VI, Art. 94.

The letter e stands for the base of the Napierian logarithms ($= 2.7182818$); $\log x$ stands for Napierian logarithm of x.

APPENDIX B.

A. Fundamental forms.—

1. $\int x^n \cdot dx = \dfrac{x^{n+1}}{n+1}$ when n is not equal to -1.

2. $\int \dfrac{dx}{x} = \log x.$

3. $\int e^x \cdot dx = e^x.$

4. $\int a^x \cdot dx = \dfrac{a^x}{\log a}.$

5. $\int \sin x \cdot dx = -\cos x.$

6. $\int \cos x \cdot dx = \sin x.$

7. $\int \tan x \cdot dx = -\log \cos x.$

8. $\int \cot x \cdot dx = \log \sin x.$

9. $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$

10. $\int \dfrac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x = \log (x + \sqrt{1+x^2}).$

11. $\int \dfrac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x = \log (x + \sqrt{x^2-1}).$

12. $\int \dfrac{dx}{1+x^2} = \tan^{-1} x.$

13. $\int \dfrac{dx}{1-x^2} = \tanh^{-1} x = \dfrac{1}{2} \log \dfrac{1+x}{1-x}.$

14. $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x.$

15. $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = \operatorname{sech}^{-1} x = -\log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right).$

16. $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}} = \operatorname{cosech}^{-1} x = -\log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right).$

17. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x.$

B. **Integrals involving trigonometric functions.**—

18. $\displaystyle\int \sec x \cdot dx = \log(\sec x + \tan x) = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$

19. $\displaystyle\int \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x) = \log\tan\frac{x}{2}.$

20. $\displaystyle\int \sin^2 x \cdot dx = \frac{x}{2} - \frac{1}{4}\sin 2x.$

21. $\displaystyle\int \cos^2 x \cdot dx = \frac{x}{2} + \frac{1}{4}\sin 2x.$

22. $\displaystyle\int \sec^2 x \cdot dx = \tan x.$

23. $\displaystyle\int \operatorname{cosec}^2 x \cdot dx = -\cot x.$

24. $\displaystyle\int \sin mx \sin nx \cdot dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$

25. $\displaystyle\int \sin mx \cos nx \cdot dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}.$

26. $\displaystyle\int \cos mx \cos nx \cdot dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$

27. $\displaystyle\int \sin^{-1} x \cdot dx = x\sin^{-1} x + \sqrt{1-x^2}.$

28. $\displaystyle\int \cos^{-1} x \cdot dx = x\cos^{-1} x - \sqrt{1-x^2}.$

C. **Integrals involving hyperbolic functions.**—

29. $\displaystyle\int \sinh x \cdot dx = \cosh x.$

30. $\int \cosh x \cdot dx = \sinh x.$

31. $\int \tanh x \cdot dx = \log \cosh x.$

32. $\int \coth x \cdot dx = \log \sinh x.$

D. **Integrals involving** $(a + bx)$.—

33. $\int \frac{dx}{a+bx} = \frac{1}{b} \log (a + bx).$

34. $\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$

35. $\int \frac{x \cdot dx}{a+bx} = \frac{1}{b^2} [a + bx - a \log (a + bx)].$

36. $\int \frac{x \cdot dx}{(a+bx)^2} = \frac{1}{b^2} \left[\log (a + bx) + \frac{a}{a+bx}\right].$

E. **Integrals involving** $(a + bx^2)$.—

37. $\int \frac{dx}{c^2+x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c}.$

38. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c} \log \frac{c+x}{c-x}.$

39. $\int \frac{x \cdot dx}{a+bx^2} = \frac{1}{2b} \log \left(x^2 + \frac{a}{b}\right).$

F. **Integrals involving** $(a + bx + cx^2)$.—For brevity let X stand for $(a + bx + cx^2)$ and let q stand for $(4ac - b^2)$.

40. $\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx+b}{\sqrt{q}}$ when q is positive.

41. $\int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}$ when q is negative.

G. **Integrals involving** $\sqrt{a+bx}$.—

42. $\int \sqrt{a+bx} \cdot dx = \frac{2}{3b} \sqrt{(a+bx)^3}.$

43. $\int x\sqrt{a+bx}\cdot dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$

44. $\int x^2\sqrt{a+bx}\cdot dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$

45. $\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$

46. $\int \frac{x\cdot dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)\sqrt{a+bx}}{3b^2}.$

47. $\int \frac{x^2\cdot dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)\sqrt{a+bx}}{15b^3}.$

H. Integrals involving $\sqrt{a^2-x^2}$.—

48. $\int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

49. $\int x\sqrt{a^2-x^2}\cdot dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3}.$

50. $\int x^2\sqrt{a^2-x^2}\cdot dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}.$

51. $\int \frac{\sqrt{a^2-x^2}}{x}\cdot dx = \sqrt{a^2-x^2}-a\log\frac{a+\sqrt{a^2-x^2}}{x}.$

52. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}.$

53. $\int \frac{x\cdot dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}.$

54. $\int \frac{x^2\cdot dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

55. $\int (a^2-x^2)^{\frac{3}{2}}\cdot dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}.$

56. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2-x^2}}.$

I′. Integrals involving $\sqrt{x^2+a^2}$.—

57. $$\int \sqrt{x^2+a^2}\cdot dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right).$$

58. $$\int x\sqrt{x^2+a^2}\cdot dx = \frac{1}{3}\sqrt{(x^2+a^2)^3}.$$

59. $$\int x^2\sqrt{x^2+a^2}\cdot dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\log\left(x+\sqrt{x^2+a^2}\right).$$

60. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right).$$

61. $$\int \frac{x\cdot dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}.$$

62. $$\int \frac{x^2\cdot dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right).$$

63. $$\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{x^2+a^2}} = \frac{1}{a}\log\frac{\sqrt{x^2+a^2}-a}{x}.$$

64. $$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}.$$

65. $$\int (x^2+a^2)^{\frac{3}{2}}\cdot dx = \frac{x}{2}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^4}{8}\log\left(x+\sqrt{x^2+a^2}\right).$$

66. $$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2+a^2}}.$$

I″. Integrals involving $\sqrt{x^2-a^2}$.—All of the forms 57 to 66 apply in this case by simply changing the sign of a^2 except form 63 which becomes:

67. $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

68. $\int \sqrt{2ax - x^2} \cdot dx = \frac{x - a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \operatorname{vers}^{-1} \frac{x}{a}.$

69. $\int x \sqrt{2ax - x^2} \cdot dx = -\frac{3a^2 + ax - 2x^2}{6} \sqrt{2ax - x^2} + \frac{a^3}{2} \operatorname{vers}^{-1} \frac{x}{a}.$

70. $\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{vers}^{-1} \frac{x}{a}.$

71. $\int \frac{x \cdot dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \operatorname{vers}^{-1} \frac{x}{a}.$

72. $\int \frac{\sqrt{2ax - x^2}}{x} \cdot dx = \sqrt{2ax - x^2} + a \operatorname{vers}^{-1} \frac{x}{a}.$

73. $\int \frac{dx}{x \sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax}.$

APPENDIX C

A SELECTED LIST OF TREATISES ON MATHEMATICS AND ON THE VARIOUS BRANCHES OF MATHEMATICAL PHYSICS.

HISTORY.

See *A History of Mathematics* by Florian Cajori, The Macmillan Co., New York, 1894

A good sketch of the history of calculus is given in the article *Infinitesimal Calculus* in the 9th edition of the *Encyclopedia Britannica.*

Sir David Brewster's *Life of Newton*, Thomas Constable & Co., Edinburgh, 1855, is well worth reading by anyone who is interested in the history of mathematics.

The theory of infinitesimals and the theory of limits have occasioned a very great deal of discussion in the past. These matters are discussed somewhat at length in the Britannica article above mentioned, but the most edifying discussion of this subject is that which is given in the introductory chapter to Augustus De Morgan's *Differential and Integral Calculus*, London, 1842.

GENERAL TREATISES.

An extremely interesting and facinating book for the student of mathematics is W. K. Clifford's* *Common Sense of the Exact Sciences,* The International Scientific Series, D. Appleton & Co., New York, 1888.

One of the best general treatises is Edouard Goursat's *Cours d'Analyse Mathematique,* two volumes, Gauthier-Villars, Paris, 1905. This valuable work has been translated into English by

* W. K. Clifford's philosophical essays (two volumes, Macmillan & Co., London, 1879) and especially his small book entitled *Seeing and Thinking* (Macmillan & Co., London, 1880) are extremely readable and interesting.

E. R. Hedrick and it is published as Goursat-Hedrick *Mathematical Analysis*, Ginn & Co., Boston.

The various mathematical articles in the ninth and eleventh editions of the *Encyclopedia Britannica* are a great help to the student of mathematics.

The great reference work in pure and applied mathematics is the *Encyclopädie der mathematischen Wissenschaften*, B. G. Teubner, Leipsig. The publication of this encyclopedia was begun in 1905 and it is not yet finished. In the meantime the work is being republished in a revised and enlarged form in a French translation.

The most extensive general treatise on mathematical physics is Winkelman's *Handbuch der Physik* in six volumes, second edition, J. A. Barth, Leipsig, 1909.

THEORY OF FUNCTIONS.

The theory of functions is one of the most facinating branches of pure mathematics.

A good discussion of this subject is given in the second volume of Goursat's *Cours d'Analyse Mathematique* (Goursat-Hedrick *Mathematical Analysis*). One of the best books on the subject is H. Durège's *Elements of the Theory of Functions* (English translation by Fisher and Schwatt), Fisher and Schwatt, Philadelphia, 1896.

Two very extensive treatises on the function theory are A. R. Forsythe's *Theory of Functions of a Complex Variable*, Cambridge University Press, 1893; and Harkness and Morley's *Theory of Functions*, Macmillan & Co., London, 1893.

DIFFERENTIAL EQUATIONS

A very good elementary treatise is D. A. Murray's *Introductory Course in Differential Equations*, Longmans Green & Co., New York, 1897. See also W. W. Johnson's *Differential Equations*, John Wiley & Sons, New York, 1890. The second volume of

Goursat-Hedrick's *Mathematical Analysis* contains a good discussion of differential equations.

A very good treatise on partial differential equations is W. E. Byerly's treatise entitled *Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics*, Ginn & Co., Boston, 1893.

An important general method of solving linear differential equations is given by P. A. Lambert in the *Annals of Mathematics*. First paper in Vol. XI, pages 185–192, July, 1910; second paper, Vol. XIII, pages 1–10, September, 1911.

QUATERNIONS AND VECTOR ANALYSIS.

(See page 211.)

HYPER-GEOMETRY AND RELATIVITY.

A very interesting development of mathematics is what is called non-Euclidean geometry or hyper-geometry. Perhaps the most interesting thing foı a beginner to read on this subject is Helmholtz's popular lecture entitled "Ueber den Ursprung und die Bedeutung der geometrischen Axiome" *Vorträge und Reden*, Vol. II, F. Vieweg & Son, Braunschweig, 1884. These very interesting lectures of Helmholtz have been translated into English, first series by Dr. Pye-Smith, second series by E. Atkinson, and both series are published by Longmans, Green & Co.

See section VI of article *Geometry* in the 11th edition of the *Encyclopedia Britannica*. A good discussion of non-Euclidean geometry is given in A. N. Whitehead's *Universal Algebra*, Cambridge University Press, 1898. See also Stäckel und Engel, *Theorie der Parallellinien von Euklid bis auf Gauss*, Leipsig, 1895; F. Klein, *Nicht Euklidische Geometrie*, Gottingen, 1893; and P. Barbarin, *La Géométrie non-Euclidienne*, Paris, 1902.

The subject of non-Euclidean geometry is closely related to a very recent development in theoretical physics called the theory or relativity. A very simple discussion of this subject is given by W. S. Franklin in an article entitled The Principle of Relativity, *Journal of the Franklin Institute*, July, 1911. A very interesting article on this subject is the eighth lecture in Max Planck's *Acht*

Volesungen über theoretische Physik, S. Hirzel, Leipsig, 1910. The following articles by G. N. Lewis, R. C. Tolman and E. B. Wilson, are interesting, and all but the last one are easy to read. *Philosophical Magazine*, Vol. 16, pages 705–717; *American Academy Proceedings*, Vol. 44, pages 711–724; *American Academy Proceedings*, Vol. 48, pages 389–507. A general treatise is M. Laue's *Das Relativitätsprinzip*, Braunschweig, F. Vieweg und Sohn, second edition, 1913.

ASTRONOMY.*

Most of the recent researches in astronomy have been in the line of what is called astro-physics, and but few additions have been made to the older theoretical astronomy with its beautifully complete mathematical theory.

Good books on theoretical astronomy for the beginner are C. A. Young's *General Astronomy*, Ginn & Co., Boston, 1898; and W. W. Campbell's *Elements of Practical Astronomy*, The Macmillan Co., New York, 1899.

Perhaps the best treatise on astronomical measurements is Wm. Chauvenet's *Spherical and Practical Astronomy*, two volumes, J. B. Lippincott, Philadelphia, 1863. For a treatise on astronomical calculations see Jas. C. Watson's *Theoretical Astronomy*, J. B. Lippincott, Philadelphia, 1867. See also *Bahnbestimmung der Kometen und Planeten*, T. R. v. Oppolzer, two volumes, Wm. Engelmann, Leipsig, 1882.

Perhaps the greatest treatise on theoretical astronomy is *Traité de Méchanique Céleste*, P. S. Laplace, five volumes, Paris, 1799. English translation by Nathaniel Bowditch, Boston, 1829.

Theoretical astronomy goes far beyond every other branch of

* Although we are here concerned chiefly with books on mathematical theory it is worth while, perhaps, to mention some of the extremely interesting descriptive books on astronomy: Simon Newcomb's *Popular Astronomy*, Harper Bros., New York, 1882. J. Norman Lockyer's *Stargazing*, Macmillan & Co., London, 1878. Richard A. Proctor's *Other Worlds than Ours*, Longmans, Green & Co., London, 1878. Also C. A. Young's *General Astronomy* is arranged so that the general reader can easily omit the more difficult parts and the book then becomes a very good descriptive treatise.

applied mathematics in the legitimate use of elaborate formulas and in the tremendously laborious numerical calculations that are involved. The student can get an idea of the former and a faint suggestion of the latter by looking over the four large volumes of *The Collected Mathematical Works* of George W. Hill which have been recently published (1905) by The Carnegie Institution of Washington and placed in every important library in the United States. Dr. Hill's greatest contribution to theoretical astronomy was a new method for calculating the motion of the moon.

PROBABILITY.

In many respects the theory of probability is the most important branch of mathematics for the experimental scientist.

The article *Probability* in the eleventh edition of the *Encyclopedia Britannica* is a very good outline of the theory of probability. See also De Morgan's treatise on probability in the *Cabinet Cyclopedia*, London, 1838. Laplace's *Theorie analytique des Probabilités*, Paris, 1820; is one of the great treatises on the subject.

The application of the theory of probability to measurement is treated in Mansfield Merriman's *Least Squares*, John Wiley & Sons, New York, eighth edition, 1903; in R. S. Woodward's *Probability and Theory of Errors*, John Wiley & Sons, New York, 1906; and in A. de F. Palmer's *Theory of Measurements*, McGraw-Hill Book Co., New York, 1912.

The application of the theory of probability to the study of statistics of all kinds and especially to biometrics (the quantitative study of variation of plants and animals) is of great importance. A simple discussion of this subject is given in the article *Variation and Selection* in the eleventh edition of the *Encyclopedia Britannica*.

The kinetic theory of gases, one of the most important branches of theoretical physics is a phase of the theory of probability. See Watson's *Kinetic Theory of Gases*, The Clarendon Press, Oxford, 1893. See also the epoch-making work, Ludwig Boltzmann's *Vorlesungen über Gastheorie*, in two parts, J. A. Barth, Leipsig, 1895 and 1898. The kinetic theory of gases is treated in a very

general way by J. Willard Gibbs in his *Statistical Mechanics*, Charles Scribner's Sons, New York, 1902.

A very good discussion of the kinetic theory of gases is given in Nernst's *Theoretical Chemistry*, Chapter II, translated by Lehfeldt, Macmillan & Co., London, 1904.

The ideas of the kinetic theory of gases are extensively used in the recent theories of the discharge of electricity through gases and in the theories of radioactivity. See especially J. J. Thomson's *Conduction of Electricity through Gases*, Cambridge University Press, 1906. See also E. Rutherford's *Radioactive Transformations*, Charles Scribner's Sons, New York, 1906.

The modern statistical theory of radiant heat is also a branch of the theory of probability. Some idea of this subject can be obtained from lecture 6 of *Acht Vorlesungen über theoretische Physik*, Max Planck, S. Hirgel, Leipsig, 1910.

THERMODYNAMICS.

There is a widespread notion that theoretical thermodynamics makes a very severe demand upon the methods of higher mathematics, whereas, as a matter of fact its demands are less perhaps than any other branch of mathematical physics.

An extremely simple development of the mathematical theory of thermodynamics is given in Franklin and MacNutt's *Mechanics and Heat*, pages 350–397, The Macmillan Co., 1910.

A very good advanced treatise is Max Planck's *Treatise on Thermodynamics* (English translation by Alexander Ogg), Longmans Green & Co., London, 1903. Another important treatise is Edgar Buckingham's *Theory of Thermodynamics*, The Macmillan Co., New York, 1900.

R. Clausius' *Mechanische Wärmetheorie*, third edition, F. Vieweg & Sons, Braunschweig, 1887, is a very important work.

The student interested in theoretical thermodynamics will find it well worth while to read the celebrated paper of J. Van't Hoff entitled "The function of osmotic pressure in the analogy between solutions and gases," *Philosophical Magazine*, Vol. XXVI, pages 81–105, August, 1888.

The best treatment of thermodynamics for the experimental chemist is that which is given in a more or less disconnected manner in W. Nernst's *Theoretical Chemistry*, English translation by R. A. Lehfeldt, Macmillan & Co., London, 1904.

A good treatment of thermodynamics for the steam engineer is that which is given in chapters II and III (pages 37–159) of J. A. Ewing's *The Steam Engine and other Heat Engines*, third edition, Cambridge University Press, 1910.

THEORETICAL MECHANICS.

There are many good beginner's treatises on theoretical mechanics. A good advanced treatise is Alexander Ziwet's *Elementary Treatise on Theoretical Mechanics*, in two parts, The Macmillan Co., New York, 1893.

Every student of mathematical physics should read a portion, at least, of Sir Isaac Newton's *Principia*.

An extremely interesting and readable book is Poinsot's *Theorie Nouvelle de la Rotation des Corps*, second edition, Paris, 1851. Harold Crabtree's *Spinning Tops*, Longmans Green & Co., London, 1909 (about), is a good mathematical treatment of the theory of the rotation of a rigid body.

The most exhaustive treatises on theoretical mechanics are: Geo. M. Minchin's *Treatise on Statics*, third edition, two volumes, The Clarendon Press, Oxford, 1886; Edward J. Routh's *Elementary Rigid Dynamics*, Macmillan & Co., London, 1860; and Edward J. Routh's *Advanced Rigid Dynamics*, Macmillan & Co., London, 1860.

See lecture 7 in *Acht Vorlesungen über theoretische Physik* by Max Planck, S. Hirzel, Leipsig, 1910. This lecture deals with the most fundamental principle of physics, the so-called principle of least action.

THEORY OF SOUND.

One of the best books on this subject for the beginner is J. H. Poynting and J. J. Thomson's *Sound*, Charles Griffin & Co., London, 1899. See also the article *Acoustics* in the 9th edition

of the *Encyclopedia Britannica.* Helmholtz's great work *The Sensations of Tone* (English translation by Alexander J. Ellis), contains a series of appendices on mathematical theory.

The most comprehensive treatise is Lord Rayleigh's *Theory of Sound* in two volumes, Macmillan & Co., London, 1877. Second edition revised and enlarged 1894.

THEORY OF ELECTRICITY AND MAGNETISM.

An understanding of vector analysis, as developed in chapter IX of this text, is absolutely necessary before one can begin the study of Maxwell's theory of electricity and magnetism.

A very simple discussion of Maxwell's theory is given in Franklin's *Electric Waves*, pages 186–196, The Macmillan Co., New York, 1909. The beginner will be greatly helped by E. Atkinson's translation of Mascart and Joubert's *Treatise on Electricity and Magnetism*, Vol. I, Thos. de la Rue & Co., London, 1883.

The great treatise on this subject is Maxwell's original treatise in two volumes entitled *Electricity and Magnetism*. The first edition of this work appeared in 1873. Third, edition very slightly altered, The Clarendon Press, 1891.

The study of Maxwell's treatise is greatly facilitated by Mascart and Joubert's treatise above mentioned; by the study of Heinrich Hertz's *Electric Waves*, translated by D. E. Jones, Macmillan & Co., London, 1893; and by the study of A. G. Webster's *Electricity and Magnetism*, Macmillan & Co., London, 1897.

A most excellent treatise for the student is Abraham and Föppl's *Theorie der Electrizität;* Vol. I, *Einführung in die Maxwellsche Theorie*, 3d edition, Leipsig, 1907; Vol. II, *Electromagneschie Theorie der Strahlung* (*Electronentheorie*), Leipsig, 1905.

Recent Researches Electricity and Magnetism by J. J. Thomson, The Clarendon Press, Oxford, 1893, contains a great deal of interest especially on the subject of electric waves. See also J. J. Thomson's *Conduction of Electricity through Gases*, Cambridge University Press, 1906. This important book deals principally with the electron theory, although, of course, the book deals almost entirely with experimental researches.

THE THEORY OF LIGHT.

One of the best books for the student is Thomas Preston's *Theory of Light*, third edition, Macmillan & Co., London, 1901.

An extremely interesting book, partly theoretical and partly descriptive, is A. A. Michelson's *Light Waves and their Uses*, University of Chicago Press, 1903. R. W. Wood's *Physical Optics*, second edition, The Macmillan Co., New York, 1910, contains a great deal of interesting and important theory.

The theory of lenses and optical instruments is treated in a very exhaustive manner by Czapski, von Rohr and Eppenstein in Winkelmann's *Handbuch der Physik*.

One of the most interesting series of original memoirs is that of Augustin Fresnel, see Fresnel's *Oevres Completes*, Vol. I, pages 1–382, Paris, 1866.

A very complete treatise on the theory of light (electromagnetic) is that of P. Drude; English translation by C. R. Mann and R. A. Millikan, Longmans Green & Co., New York, 1902.

HYDROMECHANICS.

One of the best treatises on this subject for the student is the article *Hydromechanics* in the ninth edition of the *Encyclopedia Britannica*. Part I of this article is devoted to *Hydrostatics*. In this part the problem of the figure of the earth is discussed and also the important practical problem of the stability of floating bodies. Part II of this article is devoted to *Hydrodynamics*, the highly mathematical theory of the motion of a frictionless fluid. Part III of this article is devoted to *Hydraulics* from the point of view of the experimental physicist and the engineer. Parts I and II were written by A. G. Greenhill and Part III was written by W. C. Unwin. Professor Unwin's article has been published as a separate treatise, Macmillan & Co., London, 1907.

See also G. M. Minchin's *Treatise on Hydrostatics*, two volumes, second edition, Clarendon Press, Oxford, 1912; Horace Lamb's *Hydrodynamics*, Cambridge University Press, 1879; third edition 1906; and A. B. Bassett's *Hydrodynamics*, two volumes, Deighton Bell & Co., Cambridge, 1888.

THEORY OF ELASTICITY.

An understanding of vector analysis as developed in chapter IX of this text is very helpful in the study of the theory of elasticity. Thus the ideas which are established in Art. 136 are the foundation of the elementary theory of elasticity as given in Franklin and MacNutt's *Mechanics and Heat*, pages 182–218, The Macmillan Co., New York, 1910. This is perhaps the simplest existing elementary treatise on the theory of elasticity.

The matter which is discussed in Art. 136, namely, the theory of elastic strains, is discussed in W. K. Clifford's *Kinematic*, part I, Macmillan & Co., London, 1878, pages 158–221.

One of the best treatises on the theory of elasticity is W. J. Ibbetson's *Mathematical Theory of Elasticity*, Macmillan & Co., London, 1887; see also A. E. H. Love's *Mathematical Theory of Elasticity*, two volumes, Cambridge University Press, 1892.

The greatest reference work in the theory of elasticity is Karl Pearson's *History*.

CRYSTALLOGRAPHY.

It is not generally known that one of the most interesting and remarkable branches of mathematical physics is the theory of crystallography. Thus the purely mathematical theory of regular-point-systems in space is in complete accord with experimental studies of crystal forms. A regular-point-system in space is called a *space lattice*. The purely mathematical theory of the space lattice is treated in Sohncke's *Theorie der Krystallstruktur*, B. G. Tuebner, Leipsig, 1879. Very complete treatises on crystallography are Groth's *Physikalische Krystallographie*, Wm. Englemann, Leipsig, 1895; and Liebisch's *Grundriss der Physikalischen Krystallographie*, Veit & Co., Leipsig, 1896. In both of these books the geometrical theory of crystallography is fully treated.

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